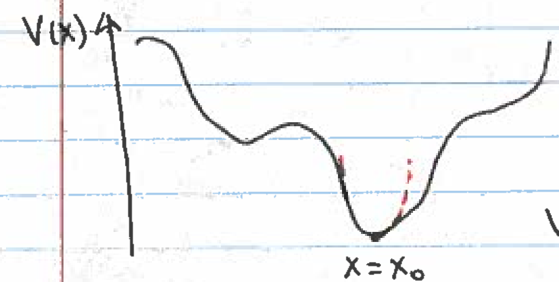
Simple Harmonic Oscillator

(almost as beloved by physisists as a spherical cow)

$$V(x) = \frac{1}{2} k x^2 = \frac{1}{2} m \omega^2 x^2$$

Why this potential is so important? In most situations it describes the motion of a system near equilibrium

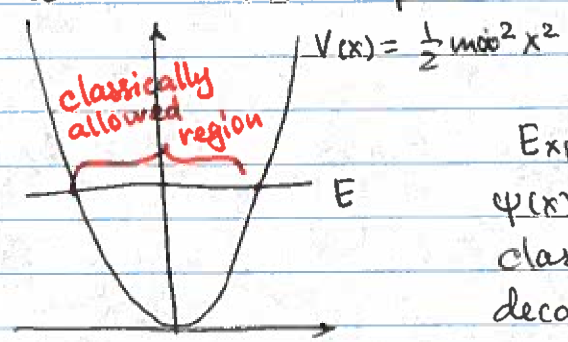


at the equilibrium

$$\frac{d}{dx} V(x = x_0) = 0$$

$$V(x) = V(x_0) + \cancel{\overbrace{\frac{dV}{dx}(x - x_0)}^{=0}} + \underbrace{\frac{1}{2}\frac{d^2V}{dx^2}(x - x_0)^2}_{\text{leading term}} + \dots$$

Spatial distribution of a particle in a harmonic potential



; $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$

Expectation: the wave function $\psi(x)$ oscillate within the classically allowed region, and decays outside

Shrodinger equation in x- basis

$$\hat{H}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \psi = E\psi$$

Solving this equation will give us eigenstates and eigen energies of this Hamiltonian, its stationary states

Steps to solve this equation

① Move to the dimentional variable

$$x \rightarrow y = \sqrt{\frac{m\omega}{\hbar}} \cdot x \qquad \mathcal{E} = \frac{2E}{\hbar\omega}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi$$

transforms into

$$\frac{d^2\psi(y)}{dy^2} + (\mathcal{E} - y^2)\psi(y) = 0$$

only one free parameter left!

② Figure out asymptotic behavior for $|y| \rightarrow \infty$ the solution must approach 0

if y is large $y^2 - \mathcal{E} \approx y^2$ (for finite $\mathcal{E}$)

@ $y \rightarrow \infty$ $\frac{d^2\psi}{dy^2} - y^2\psi(y) = 0 \Rightarrow \psi(y) = A e^{-y^2/2}$

($e^{y^2/2}$ is impossible)

③ Looking for a solution in a polynomial form (including the found asymptotic)

$$\psi(y) = h(y)\, e^{-y^2/2} \qquad \text{where} \quad h(y) = \sum_{k=0}^{N} a_k y^k$$

$$\frac{d^2\psi(y)}{dy^2} + (\mathcal{E} - y^2)\psi(y) = 0$$

transforms into

$$\frac{d^2h}{dy^2} - 2y\frac{dh}{dy} + (\mathcal{E}-1)h = 0$$

$\downarrow$

$$\sum_{k=0}^{N}\left[(k+2)(k+1)a_{k+2} - 2k a_k + (\mathcal{E}-1)a_k\right]y^k = 0$$

(4) Obtain the recurrance relationship

$$\frac{a_{k+2}}{a_k} = \frac{2k+1-\varepsilon}{(k+2)(k+1)}$$

To keep the series finite, ε can have only very specific values $\varepsilon_n = 2n+1$ (then $\frac{a_{n+2}}{a_n} = 0$, so no ~~the~~ terms with $k>n$)

$$E_n = \frac{\hbar\omega}{2}\varepsilon_n = \hbar\omega(n+\tfrac{1}{2})$$

famous equidistant energy spectrum

$h_n(y)$ – Hermite polynomials

$H_0(y) = 1$ $\quad H_2(y) = 4y^2 - 2$

$H_1(y) = 2y$ $\quad H_3(y) = 8y^3 - 12y$

Eigen functions

$$\psi_n(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right) e^{-\frac{m\omega x^2}{2\hbar}}$$

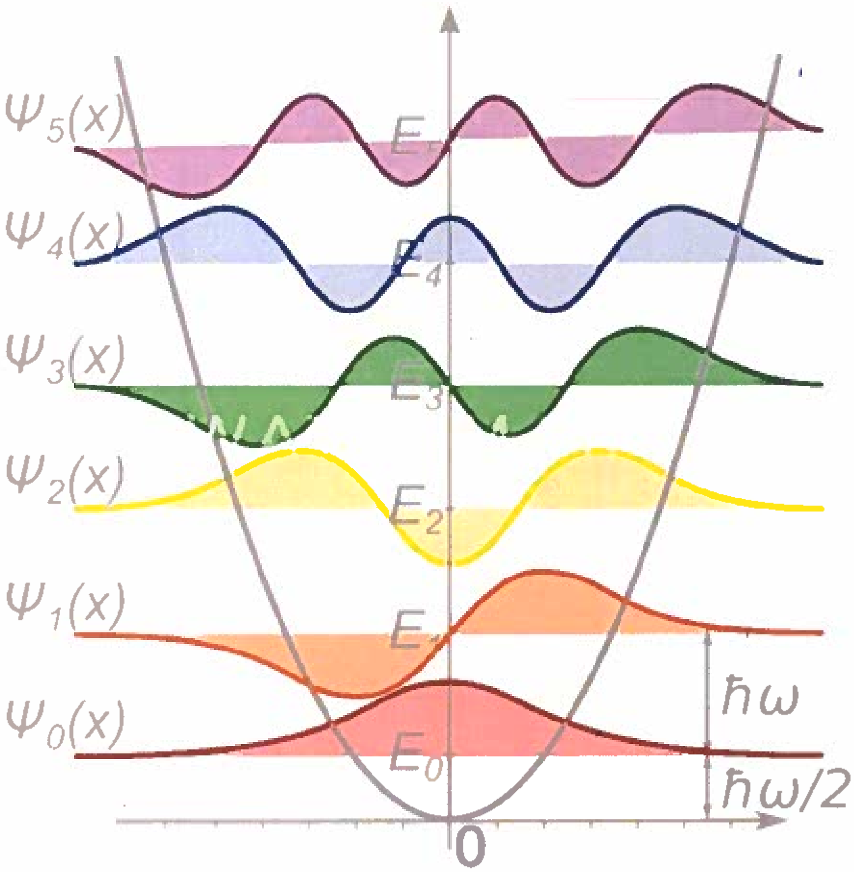
Ground state: $n=0$ $\quad E_n = \frac{1}{2}\hbar\omega$

(zero-point energy)

$$\psi_0(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}}\, e^{-\frac{m\omega x^2}{2\hbar}}$$

(gaussian distribution)

So to create a gaussian wave packet we need to trap a particle in a harmonic trap, and then let it go.



Ψ5(x)
Ψ4(x)
Ψ3(x)
Ψ2(x)
Ψ1(x)
Ψ0(x)
E4
E3
E2
E0
ħω
ħω/2
0

First four harmonic oscillator normalized wavefunctions

$$\Psi_0 = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-y^2/2}$$

$$\Psi_1 = \left(\frac{\alpha}{\pi}\right)^{1/4} \sqrt{2}y\, e^{-y^2/2}$$

$$\Psi_2 = \left(\frac{\alpha}{\pi}\right)^{1/4} \frac{1}{\sqrt{2}}(2y^2-1)\, e^{-y^2/2}$$

$$\Psi_3 = \left(\frac{\alpha}{\pi}\right)^{1/4} \frac{1}{\sqrt{3}}(2y^3-3y) e^{-y^2/2}$$

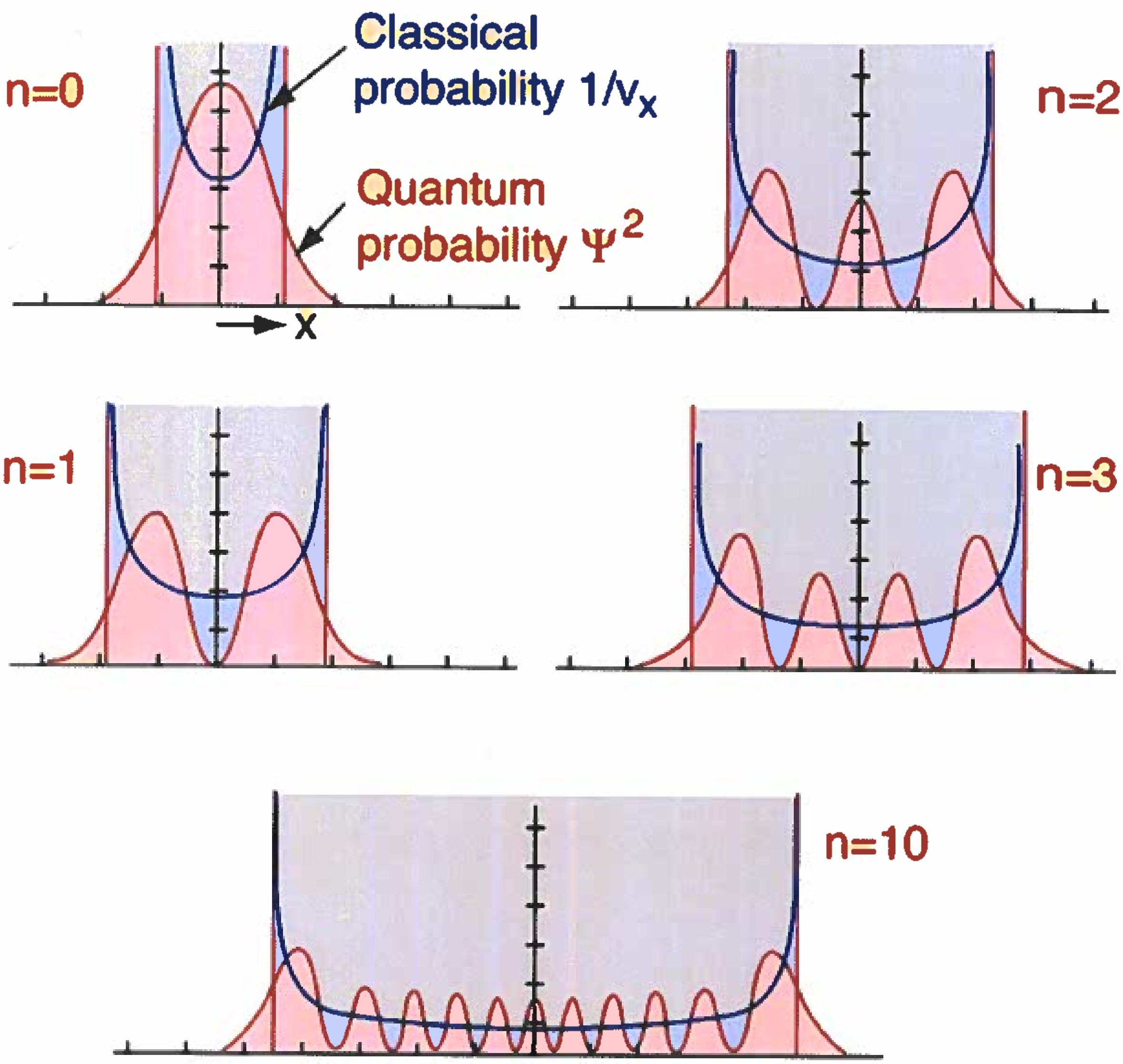
$$\alpha = \frac{m\omega}{\hbar} \qquad y = \sqrt{\alpha}\, x$$

$$\int_{-\infty}^{\infty} \exp(-ax^2)\,dx = \sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} x^n \exp(-ax^2)\,dx = \begin{Bmatrix} 0\text{: odd } n \\ \frac{\pi^{1/2}}{2a^{3/2}} : n = 2 \end{Bmatrix}$$

$$\int x^2 e^{-\alpha x^2} dx = -\frac{\partial}{\partial\alpha}\int e^{-\alpha x^2} dx = -\frac{\partial}{\partial\alpha}\sqrt{\frac{\pi}{\alpha}} =$$

$$= \frac{1}{2}\frac{\sqrt{\pi}}{\alpha^{3/2}}$$


n=0
Classical probability 1/v_x
Quantum probability Ψ^2
x
n=2
n=1
n=3
n=10