The method of calculating the time evolution of a quantum system

1. Define the hamiltonian $\hat{H}$ of the system.

2. Find eigenstates and energy eigen values for this hamiltonian

$$\hat{H}|\alpha_E\rangle = E|\alpha_E\rangle \xRightarrow{\text{all solutions}} \left\{\begin{matrix} E_1 & |1\rangle \\ E_2 & |2\rangle \\ \dots & \\ E_n & |n\rangle \\ \dots & \end{matrix}\right.$$

All eigenstates form a complite basis $\{|n\rangle\}$

3. If we need to find the time evolution of an arbitrary state:
Initial state $t=0$ $|\alpha(t=0)\rangle$ is known.
3a) Decompose this state in the hamiltonian eigenbasis $\{|n\rangle\}$

$$|\alpha(t=0)\rangle = c_1|1\rangle + c_2|2\rangle + \dots = \sum_{n=1}^{\infty (N)} c_n|n\rangle$$

3b) Use known time evolution of $\{|n\rangle\}$

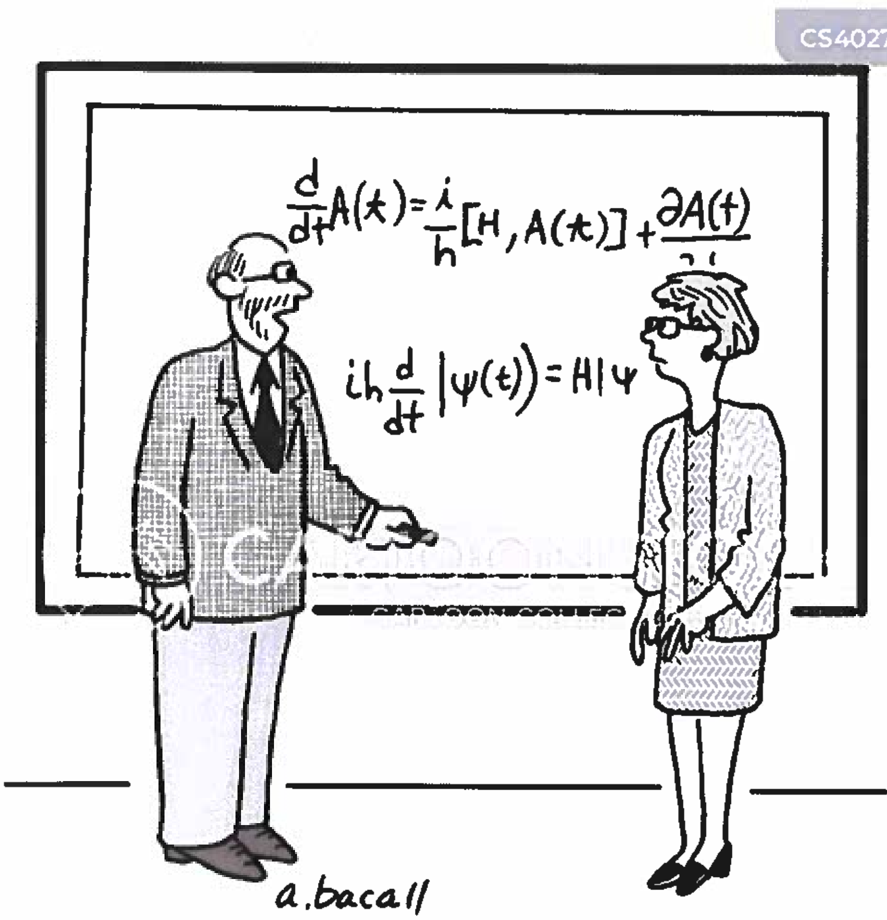
$$|\alpha(t)\rangle = c_1 e^{-iE_1t/\hbar}|1\rangle + c_2 e^{-iE_2t/\hbar}|2\rangle + \dots = \sum_{n=1}^{\infty} c_n e^{-iE_nt/\hbar}|n\rangle$$

4. If need to find the time evolution of an expectation value of an operator $\hat{A}$; we can use a differential equation

$$\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle \qquad \hat{A} \neq \hat{A}(t)$$

or explicitely calculate

$$\langle A(t)\rangle = \langle\alpha(t)|\hat{A}|\alpha(t)\rangle = \left(\sum_{n=1}^{\infty} c_n^* e^{iE_nt/\hbar}\langle n|\right)\hat{A}\left(\sum_{k=1}^{\infty} c_k e^{-iE_kt/\hbar}|k\rangle\right)$$



"I can understand Heisenberg's equation and Schrodinger's equation for quantum mechanics but I cannot understand derivative trading."

True for many physics professors
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Example 1: Spin-1/2 particle in a constant magnetic field

Potential energy of a magnetic dipole $\vec{\mu}$ in a magnetic field $\vec{B}$

$$U_{magn} = -\vec{\mu}\cdot\vec{B} = -\mu_z B_z \quad \text{if } \vec{B} = B_z \vec{k}$$

Traditionally, ~~a constant magnetic~~ we choose z-axis to be along the magnetic field

Hamiltonian (stationary spin) $\hat{H} = \hat{U} = -\hat{\mu}_z \cdot B_z = -\frac{g(-e)}{2mc}\underbrace{\hat{S}_z}_{\text{operator}} B_z$

$$\hat{H} = \left(\frac{ge}{2mc}B_z\right)\hat{S}_z = \omega_0 \hat{S}_z$$

ω_0 – larmor frequency

Step 1: $\hat{H} = \omega_0 \hat{S}_z$ Hamiltonian of a system

Step 2: eigenstates of $\hat{H}$: $|+z\rangle$ $\hat{H}|+z\rangle = \frac{\hbar\omega_0}{2}|+z\rangle$

$|-z\rangle$ $\hat{H}|-z\rangle = -\frac{\hbar\omega_0}{2}|-z\rangle$

$|\pm z\rangle$ – stationary state $\xrightarrow{t>0}$ $e^{\mp i\frac{\omega_0 t}{2}}|\pm z\rangle$

Step 3: Find time evolution of $|x\rangle$

$$|\psi(t=0)\rangle = |x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-\frac{i\omega_0 t}{2}}|+z\rangle + \frac{1}{\sqrt{2}}e^{i\omega_0 t/2}|-z\rangle$$

$t=0$ $\quad |\psi(t=0)\rangle = |+x\rangle$

$\omega_0 t = \pi$ $\quad |\psi(t=\pi/\omega_0)\rangle = \frac{1}{\sqrt{2}}e^{-i\pi/2}|+z\rangle + \frac{1}{\sqrt{2}}e^{i\pi/2}|-z\rangle =$

$$= -\frac{i}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|-z\rangle = -i\left(\frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle\right) = -i\,|-x\rangle$$

$\omega_0 t = \pi/2$ $\quad |\psi(t=\pi/2\omega_0)\rangle = \frac{1}{\sqrt{2}}e^{-i\pi/4}|+z\rangle + \frac{1}{\sqrt{2}}e^{i\pi/4}|-z\rangle =$

$$= e^{-i\pi/4}\left[\frac{1}{\sqrt{2}}|+z\rangle + e^{i\pi/2}\frac{1}{\sqrt{2}}|-z\rangle\right] = e^{-i\pi/4}\left[\frac{1}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|-z\rangle\right] =$$

$$= e^{i\pi/4}\,|+y\rangle$$

Same thing in the matrix notation

$$\hat{H} = \begin{pmatrix} \hbar\omega_0/2 & 0 \\ 0 & -\hbar\omega_0/2 \end{pmatrix}$$

Base vectors

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{t} \begin{pmatrix} e^{-i\omega_0 t/2} \\ 0 \end{pmatrix}$$

$$|-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{t} \begin{pmatrix} 0 \\ e^{i\omega_0 t/2} \end{pmatrix}$$

Any state

$$|\alpha\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \xrightarrow{t} \begin{pmatrix} c_1 e^{-i\omega_0 t/2} \\ c_2 e^{i\omega_0 t/2} \end{pmatrix}$$

Step 4: find average value of $\langle \hat{S}_x(t)\rangle$

if $|\alpha(t=0)\rangle = |x\rangle$

$$|\alpha(t)\rangle = \begin{pmatrix} 1/\sqrt{2}\, e^{-i\omega_0 t/2} \\ 1/\sqrt{2}\, e^{i\omega_0 t/2} \end{pmatrix} \qquad \langle\alpha(t)| = \left(\frac{e^{i\omega_0 t/2}}{\sqrt{2}} \quad \frac{e^{-i\omega_0 t/2}}{\sqrt{2}} \right)$$

$$\langle \hat{S}_x(t)\rangle = \langle\alpha(t)|\hat{S}_x|\alpha(t)\rangle =$$

$$= \frac{\hbar}{2}\left(\frac{e^{i\omega_0 t/2}}{\sqrt{2}} \quad \frac{e^{-i\omega_0 t/2}}{\sqrt{2}} \right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\omega_0 t/2}/\sqrt{2} \\ e^{i\omega_0 t/2}/\sqrt{2} \end{pmatrix} =$$

$$= \frac{\hbar}{2}\frac{1}{2}\left(e^{i\omega_0 t/2} \quad e^{-i\omega_0 t/2} \right)\begin{pmatrix} e^{i\omega_0 t/2} \\ e^{-i\omega_0 t/2} \end{pmatrix} = \frac{\hbar}{2}\frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t} \right)$$

$$= \frac{\hbar}{2}\cos\omega_0 t$$

Similarly

$$\langle \hat{S}_y(t)\rangle = \frac{\hbar}{2}\sin\omega_0 t$$

Thus spin rotates (precesses) ~~in~~ around z-axis with frequency ω_0 (Larmor frequency)

Same answer can be obtained from

$$\frac{d}{dt}\langle S_x(t)\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{S}_x]\rangle = \frac{i}{\hbar}\langle[\omega_0\hat{S}_z,\hat{S}_x]\rangle$$

$$[\hat{S}_z,\hat{S}_x] = i\hbar S_y$$

$$\boxed{\frac{d}{dt}\langle\hat{S}_x(t)\rangle = -\omega_0\langle\hat{S}_y(t)\rangle}$$

$$\frac{d}{dt}\langle S_y(t)\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{S}_y]\rangle = \frac{i\omega_0}{\hbar}\langle\underbrace{[\hat{S}_z,\hat{S}_y]}_{-i\hbar\hat{S}_x}\rangle = +\omega_0\langle\hat{S}_x\rangle$$

$$\boxed{\frac{d}{dt}\langle\hat{S}_y(t)\rangle = \omega_0\langle S_x(t)\rangle}$$

$$\frac{d^2}{dt}\langle\hat{S}_x(t)\rangle = -\omega_0^2\langle S_x(t)\rangle$$

solution $\langle S_x(t)\rangle = A\cos\omega_0 t + B\sin\omega_0 t$

values A, B depend on the initial conditions