Why electromagnetic field is a simple harmonic oscillator

Maxwell's equations (in vacuum)

$$\begin{cases} \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \nabla \times \vec{B} = \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} = \dfrac{1}{c^2}\dfrac{\partial \vec{E}}{\partial t} \\ \nabla \cdot \vec{E} = 0 & \nabla \cdot \vec{B} = 0 \end{cases}$$

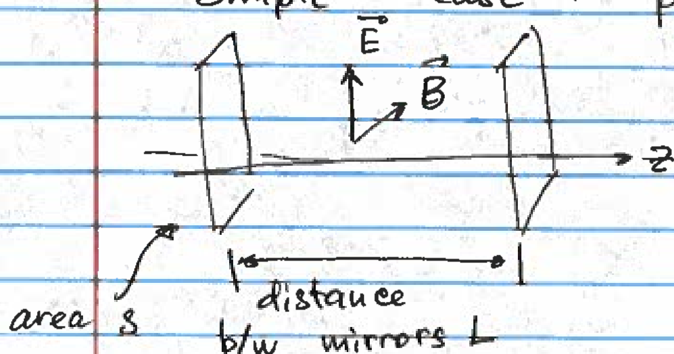
Energy of the e-m field

$$= \frac{1}{2}\int dV \left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \cancel{\approx \frac{V}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B\right)}$$

In a quantum description E, B become operators
energy $\rightarrow$ Hamiltonian

$$\hat{H}_{em} = \frac{1}{2}\int dV \left(\varepsilon_0 \hat{E}^2 + \frac{1}{\mu_0} \hat{B}^2 \right)$$

Simple case : plane e-m wave b/w two flat mirrors



$$\begin{cases} \vec{E} = E\vec{e}_x \text{ or } E\cdot\vec{i} \\ \vec{B} = B\vec{e}_y \text{ or } B\cdot\vec{j} \end{cases}$$

$$\nabla \times \vec{B} = \frac{1}{c^2}\vec{E} \rightarrow \frac{\partial B}{\partial z} = \frac{1}{c}\frac{\partial E}{\partial t}$$

wave equation

$$\frac{\partial^2 E}{\partial z^2} + \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

~~Since~~ Boundary conditions for a perfect mirror

$$E(0) = E(L) = 0$$

Solution $\quad E(z,t) = \sqrt{\dfrac{2\omega^2}{V\varepsilon_0}}\; q(t)\,\sin kz$

$$B(z,t) = -\sqrt{\frac{2}{c^2\varepsilon_0 V}}\;\dot q(t)\,\cos kz$$

Quantum case $\quad \hat E(z,t) = \sqrt{\dfrac{2\omega^2}{V\varepsilon_0}}\;\hat q(t)\,\sin kz$

$$\hat B(z,t) = -\sqrt{\frac{2}{c^2\varepsilon_0 V}}\;\dot{\hat q}(t)\,\cos kz$$

Hamiltonian

$$\hat H = \frac{1}{2}\int_V dV\left(\varepsilon_0 \hat E^2 + \frac{1}{\mu_0}\hat B^2\right) = \frac{1}{2} S\int_0^L dz\left(\varepsilon_0 \frac{2\omega^2}{V\varepsilon_0}\hat q^2 \sin^2 kz + \frac{1}{\mu_0}\frac{2}{c^2 V\varepsilon_0}\dot{\hat q}^2\cos^2 kz\right) \Rightarrow$$

$$\Rightarrow \quad \hat H = \frac{1}{2}\underset{\hat p}{\underset{\downarrow}{\dot{\hat q}}}{}^2 + \frac{1}{2}\omega^2 \underset{\hat x}{\underset{\downarrow}{\hat q}}{}^2 = \frac{1}{2}\hat p^2 + \frac{1}{2}\omega^2 \hat x^2$$

Annihilation and creation operators

$$\hat a = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat x + i\hat p) \qquad \hat a^{+} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat x - i\hat p)$$

$$\hat H = \hbar\omega\left(\hat a^{+}\hat a + \tfrac{1}{2}\right) = \hbar\omega\left(\hat n + \tfrac{1}{2}\right)$$

$$\hat E = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\hat a\, e^{-i\omega t} + \hat a^{+} e^{i\omega t}\right)\sin kz$$

$$\hat B = \frac{1}{c}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\hat a\, e^{-i\omega t} - \hat a^{+} e^{i\omega t}\right)\cos kz$$

Solution for a wave-particle duality?
E-M field is a wave in space and time, but the amount of excitation (energy) for each wave is quantized.

Quantum states of light

Number state $|n\rangle$ — known number of photons

$$\hat{H}|n\rangle = \hbar\omega(\hat{n}+\tfrac{1}{2})|n\rangle = \hbar\omega(n+\tfrac{1}{2})|n\rangle$$

$$E_n = n\cdot\hbar\omega + \underbrace{\tfrac{1}{2}\hbar\omega}_{\text{vacuum fluctuations}}$$

Very weird state!

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\langle n|\hat{E}|n\rangle = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left(\langle n|\hat{a}|n\rangle e^{-i\omega t} + \langle n|\hat{a}^+|n\rangle e^{i\omega t}\right) \times \sin kz$$

$$\langle n|\hat{E}|n\rangle = 0$$

but $\Delta E = \sqrt{\langle E^2\rangle} = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\,|\sin kz|\,\sqrt{2n+1}$

basically, if we measure electric field, all we get is fluctuations.

Closest analog of a classical e-m field is a coherent state

Eigenstate of an annihilation operator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

If $|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$

$$\hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} c_n \, \hat{a}|n\rangle = \sum_{n=\cancel{0}}^{\infty} c_n \sqrt{n}\,|n-1\rangle = \sum_{n'=0}^{\infty} c_{n'+1} \sqrt{n'+1}\,|n'\rangle$$

$$\shortparallel \quad \alpha|\alpha\rangle \quad = \quad = \alpha \sum_{n=0}^{\infty} c_n |n\rangle$$

$$c_{n}\sqrt{n+1} = \alpha c_n \qquad c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$$

$$|\alpha\rangle = \cancel{c_0}\; e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$\langle\alpha|E|\alpha\rangle = 2|\alpha| \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \cos(kz - \omega t)$$

"normal" e-m wave

Total energy $\cancel{\tfrac{1}{2}} \int \varepsilon_0 \langle E^2\rangle \, dV = |\alpha|^2 \cdot \hbar\omega$

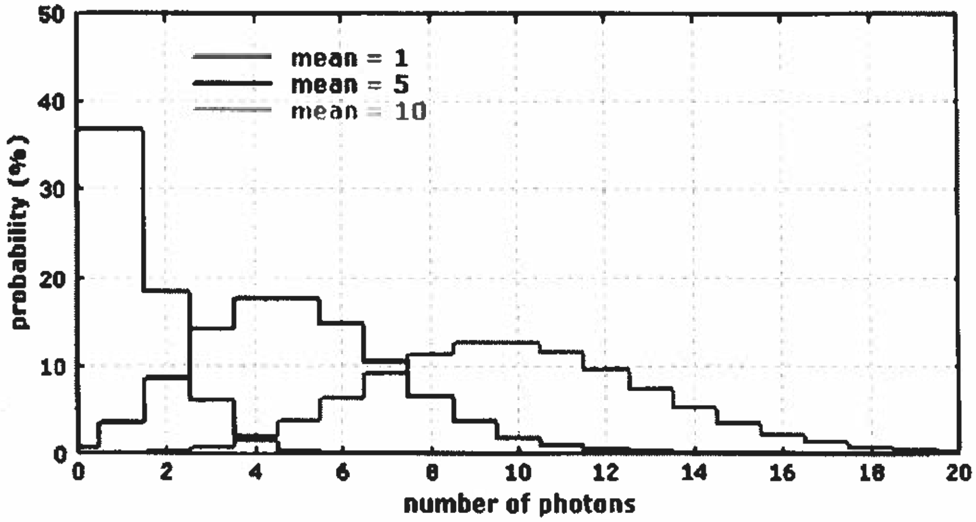
$\langle n\rangle = |\alpha|^2$

Fluctuations $\Delta n = \sqrt{\langle \hat{n}^2\rangle - \langle n\rangle^2} = |\alpha| = \sqrt{\langle n\rangle}$

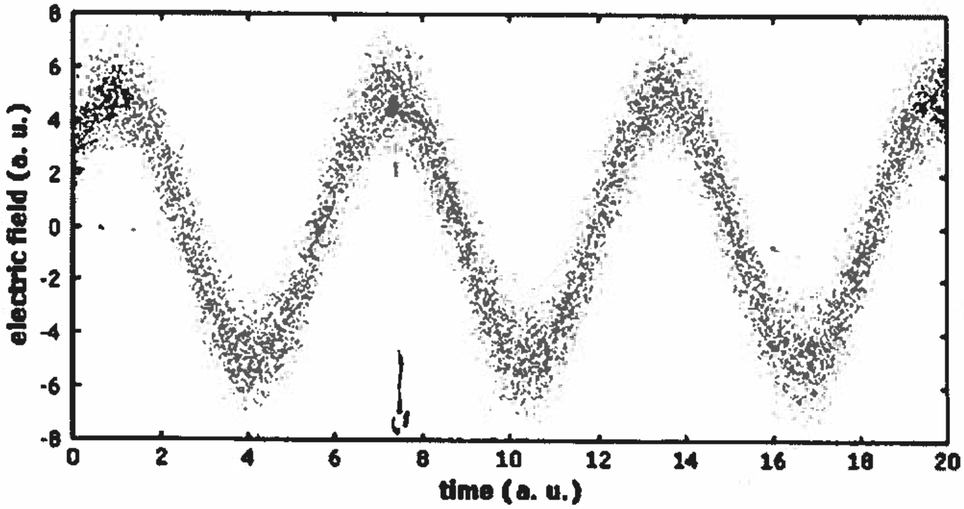
~~Pois~~ Poissonian process

$$\frac{\Delta n}{\langle n\rangle} = \frac{1}{\sqrt{\langle n\rangle}}$$

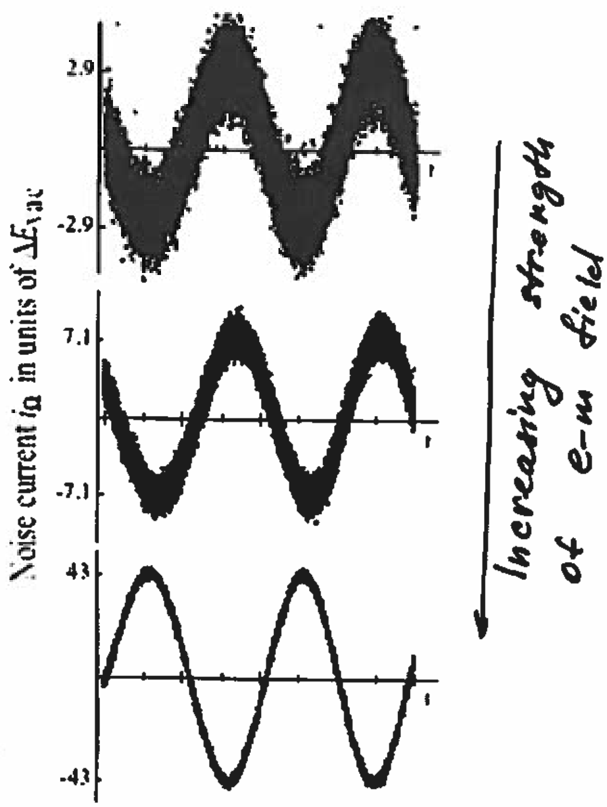
Shot noise



Photon distribution in coherent states with different mean value of photons $|\alpha|^2$



Coherent state = "fuzzy" electromagnetic wave



Since the uncertainty stays the same as amplitude grows, its effect becomes less and less noticable.