Eigen values and eigenstate of an operator

In general, an operator changes the state it acts on

$$|fin\rangle = \hat{A}\,|ini\rangle \qquad |fin\rangle \neq |ini\rangle$$

However, most operators have eigen states, that are unchanged, and the ~~value~~ average value of an operator in ~~such~~ such a state is definite $\longrightarrow$ eigen value.

$$\hat{A}\,|a_i\rangle = A_i\,|a_i\rangle \qquad a_i \text{ - possible outcomes}$$

$\hat{A}$ – operator (a matrix)

$|a_i\rangle$ – eigen state (a vector)

a_i – eigen value (a number)

Example: $\hat{J}_z$ is the operator representing the z-component of the angular momentum (for a free particle $\hat{J}_z \equiv \hat{S}_z$ – spin ang. momentum)

We already know the eigenstates and eigen values of $\hat{J}_z$:

$$\hat{J}_z\,|+z\rangle = \hbar/2\;|+z\rangle$$

$$\hat{J}_z\,|-z\rangle = \hbar/2\,|-z\rangle$$

What is the matrix of $\hat{J}_z$?

$$\hat{J}_z = \begin{pmatrix} \langle +z|\hat{J}_z|+z\rangle & \langle +z|\hat{J}_z|-z\rangle \\ \langle -z|\hat{J}_z|+z\rangle & \langle -z|\hat{J}_z|-z\rangle \end{pmatrix} = \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix} =$$

$$= \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\,\hat{\sigma}_z \qquad \boxed{\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$\hat{J}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar}{2}\;|+z\rangle\langle +z| - |-z\rangle\langle -z| =$$

$$= \frac{\hbar}{2}\left(\hat{P}_+ - \hat{P}_-\right)$$

What about $\hat{J}_x$ and $\hat{J}_y$?

$$\hat{J}_x |+x\rangle = \frac{\hbar}{2}|+x\rangle \qquad \hat{J}_x |-x\rangle = -\frac{\hbar}{2}|-x\rangle$$

In the z-basis we can calculate the matrix for J_x by evaluating the matrix elements

$$\langle +z|\hat{J}_x|+z\rangle = \langle +z|\hat{J}_x\left(\frac{1}{\sqrt{2}}|+x\rangle + \frac{1}{\sqrt{2}}|-x\rangle\right) =$$

$$= \langle +z| \frac{1}{\sqrt{2}}\frac{\hbar}{2}|+x\rangle + \frac{1}{\sqrt{2}}\left(-\frac{\hbar}{2}\right)|-x\rangle =$$

$$= \frac{\hbar}{2}\langle +z|\left(\frac{1}{\sqrt{2}}|+x\rangle - \frac{1}{\sqrt{2}}|-x\rangle\right) = \frac{\hbar}{2}\langle +z|-z\rangle = 0$$

Similarly $\langle -z|\hat{J}_x|-z\rangle = 0$

$$\langle +z|\hat{J}_x|-z\rangle = \langle +z|\hat{J}_x\left(\frac{1}{\sqrt{2}}|+x\rangle - \frac{1}{\sqrt{2}}|-x\rangle\right) =$$

$$= \langle +z|\left(\frac{1}{\sqrt{2}}\frac{\hbar}{2}|+x\rangle - \frac{1}{\sqrt{2}}\left(-\frac{\hbar}{2}\right)|-x\rangle\right) = \frac{\hbar}{2}\langle +z|\left(\frac{1}{\sqrt{2}}|+x\rangle + \frac{1}{\sqrt{2}}|-x\rangle\right)$$

$$= \hbar/2 \qquad \langle -z|\hat{J}_x|+z\rangle = \langle +z|\hat{J}_z|-z\rangle^* = \hbar/2$$

$$\hat{J}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\hat{\sigma}_x \qquad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Following the same steps

$$\hat{J}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \frac{\hbar}{2}\hat{\sigma}_y \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

<u>Pauli matrices</u>

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$\hat{\sigma}_z^2 = \hat{\sigma}_x^2 = \hat{\sigma}_y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{1}$$

More quantum operator properties

$|fin\rangle = \hat{A}\,|ini\rangle$ | $\langle fin| = \langle ini|\,\underline{\underline{\hat{A}^{+}}}$

adjoint operator

$\hat{A}^{+}$ dagger (not plus!)

$$\langle\alpha|\underbrace{\hat{A}|\beta\rangle}_{|fin\rangle}^{*} = \langle fin|\alpha\rangle = \langle\beta|\hat{A}^{+}|\alpha\rangle$$

$$A^{*}_{\alpha\beta} = A^{+}_{\beta\alpha}$$

The matrix corresponding to an adjoint operator $\hat{A}^{+}$ is transposed and complex conjugate matrix of the operator $\hat{A}$

Hermitian operators: describe reversable state evolution and physically ~~are~~ possible operation

$|fin\rangle = A\,|ini\rangle = A^{+}\,|ini\rangle \qquad \hat{A} = \hat{A}^{+}$

$$\langle\alpha|\hat{A}|\beta\rangle^{*} = \langle\beta|\hat{A}^{+}|\alpha\rangle = \langle\beta|\hat{A}|\alpha\rangle$$

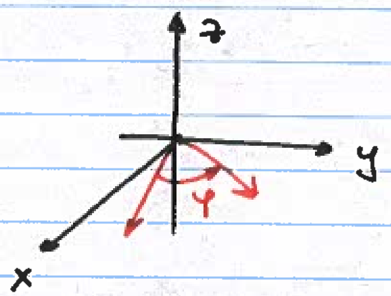
The diagonal elements of a Hermitian operator matrix are real numbers

The eigen values of a Hermitian operator are also real (hence, possible measurement outcomes)

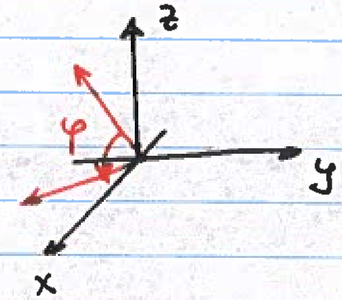
The expectation value of an operator $\hat{A}$ in the state $|\alpha\rangle$ is

$$\langle A\rangle_{|\alpha\rangle} = \langle\alpha|\hat{A}|\alpha\rangle \quad \text{(in any basis)}$$

Rotation operators



Rotation in the x-y plane
→ around z-axis



Rotation in the x-z plane
→ around y-axis

Rotation operator general notation $\hat{R}(\varphi \vec{n})$

angle of rotation — axis of rotation

We can transform ~~the state vectors~~ quantum states using the rotation operators. For example, to rotate a state in x-y plane, we need to rotate it around $\vec{k}$ (z) axis

$$|fin\rangle = \hat{R}(\varphi \vec{k})|ini\rangle = e^{-i\hat{J}_z \cdot \varphi/\hbar}|ini\rangle$$

$e^{-i\hat{J}_z\varphi/\hbar}$ can be calculated using Taylor expansion

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

$$e^{-\frac{i\varphi}{\hbar}\hat{J}_z} = \hat{\mathbb{1}} + \left(-\frac{i\varphi}{\hbar}\right)\hat{J}_z + \frac{1}{2!}\left(-\frac{i\varphi}{\hbar}\right)^2\hat{J}_z^2 + \frac{1}{3}\left(-\frac{i\varphi}{\hbar}\right)^3\hat{J}_z^3 + \dots$$

$$= \hat{\mathbb{1}} - i\varphi/_2\,\hat{\sigma}_z - \frac{1}{2!}\varphi^2/_4\,\hat{\sigma}_z^2 - i\varphi^3/_8\,\hat{\sigma}_z^3 + \frac{1}{4!}\varphi^4/_{16}\,\hat{\sigma}_z^4 - \dots$$

This can actually be simplified further, knowing that $\hat{\sigma}_z^2 = \hat{\mathbb{1}}$, $\hat{\sigma}_z^3 = \hat{\sigma}_z$, etc.

$$e^{-\frac{i\varphi}{\hbar}\hat{J}_z} = \cos(\varphi/2)\hat{\mathbb{1}} - i\sin(\varphi/2)\hat{\sigma}_z$$

$$\hat{R}(\varphi\hat{k})|+z\rangle = e^{-i\hat{J}_z\varphi/\hbar}|+z\rangle = e^{-i\,\hbar/2\cdot\varphi/\hbar}|+z\rangle = e^{-i\varphi/2}|+z\rangle$$

This rotation doesn't change the state, but adds an extra phase