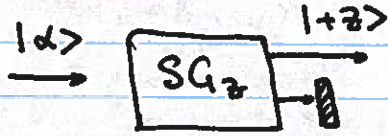
Quantum operator

Any variations or manipulations or measurements acting on a quantum state are described by an <u>operator</u>

Notation $\hat{A}$ ← hat indicates an operator

$|final\rangle = \hat{A}\,|initial\rangle$: an operator acts on an initial quant state $|initial\rangle$, and this state changes into a new, final, state $|final\rangle$

Example

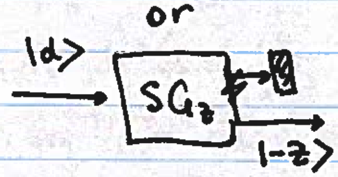


$|initial\rangle = |\alpha\rangle$

$|final\rangle = |\pm z\rangle$

Projection operator $\hat{P}_{\pm}$

or



$\hat{P}_+ |\alpha\rangle \rightarrow |+z\rangle$

$\hat{P}_- |\alpha\rangle \rightarrow |-z\rangle$

Outer or tensor product $|\alpha\rangle\langle\alpha|$

$$|\alpha\rangle\langle\alpha| = \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \begin{pmatrix} c_+^* & c_-^* \end{pmatrix} = \begin{pmatrix} |c_+|^2 & c_+ c_-^* \\ c_-^* & |c_-|^2 \end{pmatrix}$$

$$|+z\rangle\langle+z| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|-z\rangle\langle-z| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|+z\rangle\langle+z| + |-z\rangle\langle-z| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{\mathbb{1}}$$

identity

$$\hat{\mathbb{1}}|\alpha\rangle = |\alpha\rangle \iff \left(|+z\rangle\langle+z| + |-z\rangle\langle-z|\right)|\alpha\rangle = |\alpha\rangle$$

$$|+z\rangle\underbrace{\left(\langle+z|\alpha\rangle\right)}_{c_+} + |-z\rangle\underbrace{\left(\langle-z|\alpha\rangle\right)}_{c_-} = c_+|+z\rangle + c_-|-z\rangle \text{''}$$

$\hat{P}_+ = |+z\rangle\langle +z|$ $\qquad$ $\hat{P}_+ |\alpha\rangle = |+z\rangle\langle +z|\alpha\rangle = c_+ |+z\rangle$

$\hat{P}_- = |-z\rangle\langle -z|$ $\qquad$ $\hat{P}_- |\alpha\rangle = |-z\rangle\langle -z|\alpha\rangle = c_- |-z\rangle$

$\hat{P}_+$ and $\hat{P}_-$ may reduce the number of particles, hence the output amplitudes may be less than 1.

The operators can be multiplied

$$|ini\rangle \longrightarrow \boxed{A_1} \xrightarrow{|int\rangle} \boxed{A_2} \longrightarrow |fin\rangle$$

$$|fin\rangle = \hat{A}_2 \underbrace{\left[\hat{A}_1 |ini\rangle\right]}_{|int\rangle}$$

The correct order of operations is critical, since changing the order of operation may result in a different final quantum state

$$\xrightarrow{|+x\rangle} \boxed{SG_z} \xrightarrow{|+z\rangle} \boxed{SG_x} \xrightarrow{|+x\rangle} \text{output}$$

$$\hat{P}_{+z} = |+z\rangle\langle +z| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{P}_{+x} = |+x\rangle\langle +x| = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$|output\rangle = \hat{P}_{+x} \hat{P}_{+z} |+x\rangle = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \underline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}} =$$

$$= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2\sqrt{2} \\ 1/2\sqrt{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} |+x\rangle$$

in 25% cases.

wrong order

$$\hat{P}_{+z} \underbrace{\hat{P}_{+x} |+x\rangle}_{|+x\rangle} = \hat{P}_{+z} |+x\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} |+z\rangle$$

Any operator can be presented as a matrix, in any basis.

For example, in $|\pm z\rangle$ basis

$$|ini\rangle = c_+ |+z\rangle + c_- |-z\rangle$$
$$|fin\rangle = f_+ |+z\rangle + f_- |-z\rangle$$

$$\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \begin{pmatrix} A_{++} c_+ + A_{+-} c_- \\ A_{-+} c_+ + A_{--} c_- \end{pmatrix}$$

on the other hand $f_+ = \langle +z | fin\rangle = \langle +z | \hat{A} | ini\rangle$

$f_- = \langle -z | fin\rangle = \langle -z | \hat{A} | ini\rangle$

$$\hat{A}|ini\rangle = \hat{A}\left(c_+ |+z\rangle + c_- |-z\rangle\right) = c_+ \hat{A}|+z\rangle + c_- \hat{A}|-z\rangle$$

$$f_+ = \langle +z | \hat{A} | ini\rangle = c_+ \langle +z | \hat{A} | +z\rangle + c_- \langle +z | \hat{A} | -z\rangle$$

$$f_- = \langle -z | \hat{A} | ini\rangle = c_+ \langle -z | \hat{A} | +z\rangle + c_- \langle -z | \hat{A} | -z\rangle$$

$$\hat{A} = \begin{pmatrix} \langle +z | \hat{A} | +z\rangle & \langle +z | \hat{A} | -z\rangle \\ \langle -z | \hat{A} | +z\rangle & \langle -z | \hat{A} | -z\rangle \end{pmatrix}$$

Matrix element of an operator

$$A_{\alpha\beta} = \langle \alpha | \hat{A} | \beta\rangle$$