Raising & lowering operator

Brief reminder of a new notation for spin states $|s, m\rangle$ for $|j, m\rangle$

In this notation we are "locked" to z-basis

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 \quad \text{or} \quad \hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$$

$$\hat{J}^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle$$

$$\hat{J}_z |j, m\rangle = \hbar m |j, m\rangle$$

Spin-1/2 particle $\quad j = s = 1/2, \quad m = \pm 1/2$

Spin-1 particle $\quad j = s = 1, \quad m = 0, \pm 1$

Eigenstates of $\hat{J}^2$ and $\hat{J}_z$:

$$|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\hat{S}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \hat{S}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\hat{S}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

If $|y_{0,\pm1}\rangle$ are the eigenstates of $\hat{S}_y$

$$\hat{S}_y |y_0\rangle = 0 \qquad \hat{S}_y |y_{\pm1}\rangle = \pm\hbar |y_{\pm1}\rangle \leftarrow \text{true in any basis}$$

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} |y_{\pm1}\rangle = \pm\hbar |y_{\pm1}\rangle \longrightarrow \text{find } |y_{\pm1}\rangle \text{ in z-basis}$$

$$|y_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\+1\end{pmatrix} \qquad |y_{+1}\rangle = \frac{1}{2}\begin{pmatrix}1\\i\sqrt{2}\\-1\end{pmatrix} \qquad |y_{-1}\rangle = \frac{1}{2}\begin{pmatrix}1\\-i\sqrt{2}\\-1\end{pmatrix}$$

$\cancel{Ka}\,|y_0\rangle \longrightarrow \boxed{SG_z}$ → probability $1/2$, 0, $1/2$

$|y_{\pm1}\rangle \longrightarrow \boxed{SG_z}$ → probability $1/4$, $1/2$, $1/4$

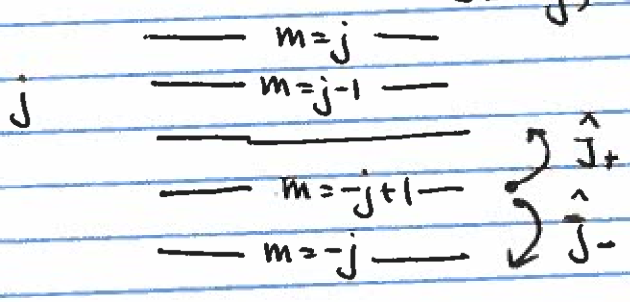
Extra family members in $\hat{J}_x, \hat{J}_y, \hat{J}_z, \hat{J}^2$ family

$\hat{J}_+ = \hat{J}_x + i\hat{J}_y$ raising operator

$\hat{J}_- = \hat{J}_x - i\hat{J}_y$ lowering operator

In general $\hat{J}_+ |j,m\rangle \propto |j,m+1\rangle$

$\hat{J}_- |j,m\rangle \propto |j,m-1\rangle$



also called ladder operators

Spin $1/2$

$\hat{J}_+ |1/2, 1/2\rangle = 0$

$\hat{J}_- |1/2, -1/2\rangle = \text{const}\cdot |1/2, 1/2\rangle$

$\hat{J}_+ |1/2, -1/2\rangle = \text{const}\, |1/2, 1/2\rangle$

$\hat{J}_- |1/2, -1/2\rangle = 0$

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y = \frac{\hbar}{2}\left[\begin{pmatrix}0&1\\1&0\end{pmatrix} \pm i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right]$$

$$\hat{J}_+ = \frac{\hbar}{\cancel{2}}\begin{pmatrix}0&1\\0&0\end{pmatrix} \qquad \hat{J}_- = \frac{\hbar}{\cancel{2}}\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

Raising & lowering operators

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ +i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \sqrt{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{J}_- = \hat{J}_x - i\hat{J}_y = \sqrt{2}\hbar\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

As in the case of spin-$1/2$ particle, $\hat{J}_+$ action will convert a state with m into a state $m+1$, and $\hat{J}_-$ — from m to $m-1$

$$J_+|1,-1\rangle = \sqrt{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \sqrt{2}\hbar\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \sqrt{2}\hbar\,|1,0\rangle$$

$$J_+|1,0\rangle = \sqrt{2}\hbar\,|1,1\rangle$$

$$J_+|1,1\rangle = \sqrt{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$\hat{J}_+|1,1\rangle = 0$ since $m=1$ is max.

Similarly $\hat{J}_-|1,1\rangle = \sqrt{2}\hbar\,|1,0\rangle$, $\hat{J}_-|1,0\rangle = \sqrt{2}\hbar\,|1,-1\rangle$

and $\hat{J}_-|1,-1\rangle = 0$

These behavior is universal, and occures for any j ~~[illegible]~~ triplets of operators that follow the same commutation relationships

How can we prove that $\hat{J}_+ |j,m\rangle \propto |j,m+1\rangle$?

$$[\hat{J}_z, \hat{J}_\pm] = [\hat{J}_z, \hat{J}_x \pm i\hat{J}_y] = [\hat{J}_z, \hat{J}_x] \pm i[\hat{J}_z, \hat{J}_y] =$$

$$= i\hbar J_y \pm i(-i\hbar \hat{J}_x) = i\hbar J_y \pm \hbar \hat{J}_x = \pm\hbar(\hat{J}_x \pm i\hat{J}_y) = \pm\hbar\hat{J}_\pm$$

Thus, if $\hat{J}_z |j,m\rangle = \hbar m |j,m\rangle$, $\hat{J}_z\hat{J}_+ - \hat{J}_+\hat{J}_z = \hbar\hat{J}_+$

Then $\hat{J}_z\hat{J}_+|j,m\rangle = (\hat{J}_+\hat{J}_z + \hbar\hat{J}_+)|j,m\rangle =$

$$= \hat{J}_+ \underbrace{[\hat{J}_z|j,m\rangle]}_{\hbar m|j,m\rangle} + \hbar\hat{J}_+|j,m\rangle = \hat{J}_+ \hbar(m+1)|j,m\rangle$$

So $\hat{J}_z[\hat{J}_+|j,m\rangle] = \hbar(m+1)[\hat{J}_+|j,m\rangle]$

must be identical to

$$\hat{J}_z|j,m+1\rangle = \hbar(m+1)|j,m+1\rangle$$

Thus $\hat{J}_+|j,m\rangle \propto |j,m+1\rangle$

Now let's assume that

$\hat{J}_+|j,m\rangle = c_{jm}\hbar|j,m+1\rangle$ and find c_{jm}

$$\langle j,m|\hat{J}_-\hat{J}_+|j,m\rangle = \langle j,m|\hat{J}_+^\dagger\hat{J}_+|j,m\rangle =$$

$$= |c_{jm}|^2\hbar^2\langle j,m+1|j,m+1\rangle = |c_{jm}|^2\cdot\hbar^2$$

At the same time $\hat{J}_-\hat{J}_+ = (\hat{J}_x - i\hat{J}_y)(\hat{J}_x + i\hat{J}_y) =$

$$= \hat{J}_x^2 + \hat{J}_y^2 + i(\hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x) = \hat{J}^2 - \hat{J}_z^2 + i\cdot i\hbar\hat{J}_z =$$

$$= \hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z$$

$$\langle j,m|\ \hat{J}^2-\hat{J}_z^2-\hbar\hat{J}_z|j,m\rangle = \hbar^2 j(j+1)-\hbar^2m^2-\hbar^2 m$$

$$|c_{jm}|^2\cdot \cancel{\hbar^2} = \cancel{\hbar^2}\,(j(j+1)-m(m+1))$$

since $|c_{jm}|^2 \geq 0$ ~~[illegible]~~ $-j-1 \leq \boxed{m \leq j}$

$$c_{jm} = \sqrt{j(j+1)-m(m+1)}$$

$$\hat{J}_+|j,m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}\ |j,m+1\rangle$$

automatically get $\hat{J}_+|j,j\rangle = 0$ $\quad m_{max}=j$

We can repeat the steps for

$$\hat{J}_-|j,m\rangle = d_{jm}\cancel{\hbar}\,|j,m-1\rangle$$

and $\langle j,m|\ \hat{J}_+\hat{J}_-|j,m\rangle = \langle j,m|\ \hat{J}^2-\hat{J}_z^2+\hbar\hat{J}_z|j,m\rangle$

$$=\hbar^2(j(j+1)-m(m-1))$$

since $|d_{jm}|^2\geq 0$ $\quad \boxed{-j \leq m}\leq j+1$

$$\hat{J}_-|j,m\rangle = \hbar\sqrt{j(j+1)-m(m-1)}\ |j,m-1\rangle$$

and $-m \leq j \leq m$

Spin $-1/2$ $\quad j=1/2 \quad m=\pm\frac{1}{2}$

Spin -1 $\quad j=1 \quad m=0,\pm 1$

Also, since we must be able to "step" from minimum $m=-j$ to max $m=j$ in integer number of steps $=2j$, then j ~~must~~ be either integer, or half-integer