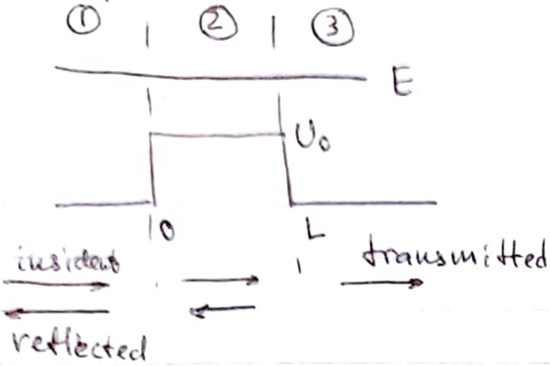
Quantum interference with quantum wave



What is the transmission probability for a particle with $E > U_0$?

Region 1 & 3 $\quad \dfrac{p_0^2}{2m} = E \quad k_0 = \dfrac{\sqrt{2mE}}{\hbar}$

$x<0, x>L$

Region 2: $\quad \dfrac{p^2}{2m} + U_0 = E \quad k = \dfrac{\sqrt{2m(E-U_0)}}{\hbar}$

$0<x<L$

$$\psi(x) = \begin{cases} A e^{ik_0x} + B e^{-ik_0x} & x<0 \\ C e^{ikx} + D e^{-ikx} & 0<x<L \\ F e^{ik_0(x-L)} & x>L \end{cases}$$

Boundary condition

@ $x=0$ $\quad \psi(0-0) = \psi(0+0) \qquad A+B = C+D$

$\psi'(-0) = \psi'(+0) \qquad ik_0A - ik_0B = ikC - ikD$

@ $x=L$ $\quad \psi(L-0) = \psi(L+0) \qquad Ce^{ikL} + De^{-ikL} = F$

$\psi'(L-0) = \psi'(L+0) \qquad ikCe^{ikL} - ikDe^{-ikL} = ik_0F$

Transmission coefficient $\quad T = \left|\dfrac{F}{A}\right|^2$ (can do, since particle move with the same speed)

$2k_0A = (k_0+k)C + (k_0-k)D$

$2kCe^{ikL} = (k+k_0)F$

$2kDe^{-ikL} = (k-k_0)F$

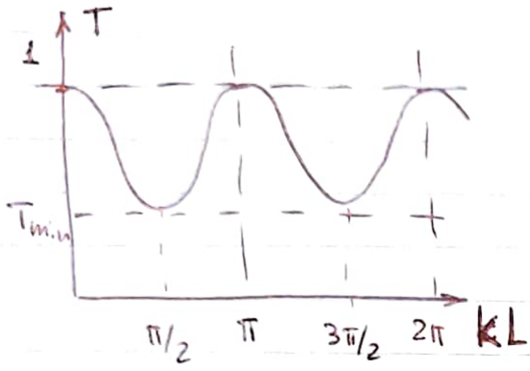
$C = Fe^{-ikL}\dfrac{k+k_0}{2k} \qquad D = Fe^{+ikL}\dfrac{k-k_0}{2k}$

$4kk_0A = (k+k_0)^2 e^{-ikL} F + (k-k_0)^2 e^{ikL} F =$

$= \Big[(k^2+k_0^2)\underbrace{(e^{-ikL} - e^{ikL})}_{-2i\sin kL} + 2kk_0\underbrace{(e^{ikL} + e^{-ikL})}_{2\cos kL}\Big] F$

$A = F\left[\cos kL - i\dfrac{k^2+k_0^2}{2kk_0}\sin kL\right]$

$$T = \left|\frac{F}{A}\right|^2 = \frac{1}{\cos^2 kL + \left(\frac{k_0^2+k^2}{2kk_0}\right)^2 \sin^2 kL} = \frac{1}{1 + \left(\frac{k^2-k_0^2}{4kk_0}\right)^2 \sin^2 kL}$$



$T_{max} = 1 \quad \sin kL = 0$

$\cos kL = 1$

$$k_n L = \pi n \Rightarrow E_n = \frac{\pi^2 n^2 \hbar^2}{2mL^2}$$

$T_{min} \longrightarrow \sin kL = 1$

$$T_{min} = \frac{4k^2k_0^2}{(k_0^2+k^2)^2}$$

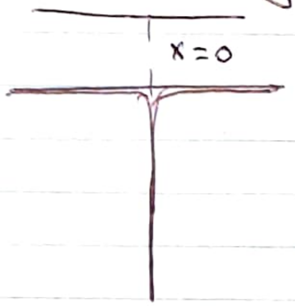
If the length of the barrier or the particle energy is just right, the reflections cancell each other out, and all the wave is transmitted without reflection

Dirac delta function

$$\delta(x) = \begin{cases} \infty & \text{at } x = 0 \\ 0 & x \neq 0 \end{cases} \qquad \delta(x-a) = \begin{cases} \infty & x = a \\ 0 & x \neq a \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(x-a)\, dx = 1 \qquad \int_{-\infty}^{+\infty} f(x)\, \delta(x-a)\, dx = f(a)$$

Scattering off δ- function well



$$U(x) = -U_0\, \delta(x)$$

$$\psi(x) = \begin{cases} A e^{ik_0 x} + B e^{-ik_0 x} & x < 0 \\ C e^{ik_0 x} & x > 0 \end{cases}$$

Wave function is still continuous

$$A + B = C$$

However, it is not smooth @ $x = 0$ ~~since~~ since $U(x=0) \Rightarrow \infty$

Shrodinger equation

$$\lim_{\Delta x \to 0} \left\{ \int_{-\Delta x}^{\Delta x} -\frac{\hbar^2}{2m} \psi''(x)\, dx + \int_{-\Delta x}^{\Delta x} (-U_0)\, \delta(x)\, \psi(x)\, dx = \int_{-\Delta x}^{\Delta x} E\psi(x)\, dx \right\}$$

$$-\frac{\hbar^2}{2m} \left[\psi'(+0) - \psi'(-0) \right] - U_0 \psi(0) = 0$$

$$-\frac{\hbar^2}{2m} \left[ik_0 A - ik_0 B - ik_0 C \right] - U_0 (A+B) = 0$$

$$-\frac{i\hbar^2 k_0}{2m} \left[\cancel{A} - B - \cancel{A} - B \right] = U_0 (A+B)$$

$$\left(\frac{i\hbar^2 k_0}{2m} + U_0 \right) B = U_0 A \quad \Rightarrow \quad \frac{B}{A} = \frac{U_0}{\frac{i\hbar^2 k_0}{2m} - U_0}$$

$$R = \frac{U_0^2}{\hbar^4 k_0^2 / 4m^2 + U_0^2} \qquad T = \frac{\hbar^4 k_0^2 / 4m^2}{\hbar^4 k_0^2 / 4m^2 + U_0^2}$$