Motion of Quantum Particles in flat potentials

Reminder: free-moving particle of momentum p_x is described in x-representation as a plane wave

$$\psi_{p_x}(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ip_x \cdot x/\hbar}$$

This is an eigenfunction of the momentum operator

$$\hat{p}_x \psi_{p_x}(x) = -i\hbar \frac{\partial}{\partial x} \psi_{p_x}(x) = p_x \cdot \psi_{p_x}(x)$$

It is also an eigenstate of a Hamiltonian with flat potential $\hat{U} = U_0$

$$\hat{H} = \hat{K} + \hat{U} = \frac{\hat{p}_x^2}{2m} + U_0$$

$$\left(\frac{\hat{p}_x^2}{2m} + U_0\right)\psi_{p_x}(x) = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_{p_x}(x)}{\partial x^2} + U_0 \psi_{p_x}(x) = E\psi_{p_x}(x)\right.$$

$$E = \frac{p_x^2}{2m} + U_0 \Rightarrow p_x = \sqrt{2m(E-U_0)}$$

De Broigle wavelength

$$\psi_{p_x}(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ip_x \cdot x/\hbar}$$

periodic in space: $\frac{p_x \cdot x}{\hbar} = 2\pi, 4\pi, \ldots$

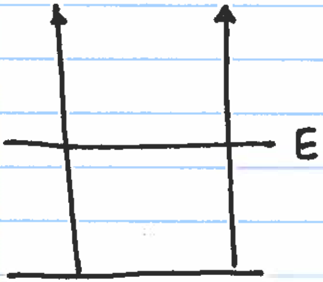
Spatial oscillations with wavelength

$$\lambda = \frac{2\pi\hbar}{p_x}$$

Wave-particle duality: a free particle of given energy ~~is~~ behaves as a wave when subject of interference or diffraction, but it behave as a particle when

Our plans for quantum systems with flat potentials

Infinite square well



1. Very convenient simplest model for any quantum system with confinement

Finite square well (more realistic)

1. Nuclea model
2. Electron quantization

The behavior of such a quantum particle is very similar to behavior of an ideal electromagnetic plane wave of given frequency. A laser beam is a good approximation

Quantum particle

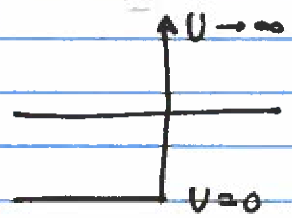
$$\psi_{p_x}(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{i p_x \cdot x}{\hbar} - \frac{i E \cdot t}{\hbar}}$$

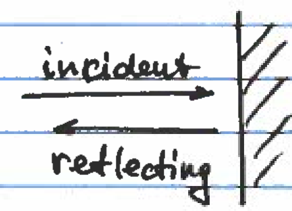
Running laser beam of a wavevector $k_x = \frac{2\pi}{\lambda}$

$$E(x,t) = E_0 \, e^{i k_x t - i\omega t}$$

Optic analogs of flat potential energy

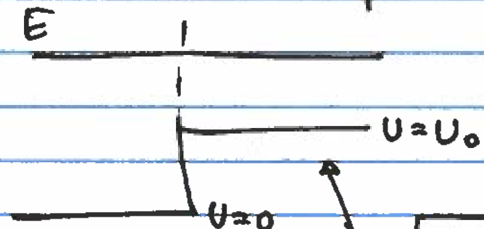
Infinite wall | Optics

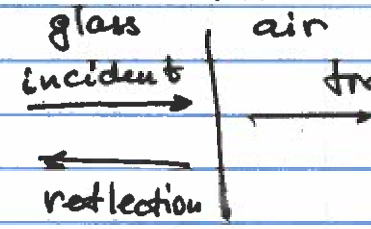




perfect mirror

Potential step $E > U_0$ | Two trasparent materials





$p = \sqrt{2m(E-U_0)}$

λ_0

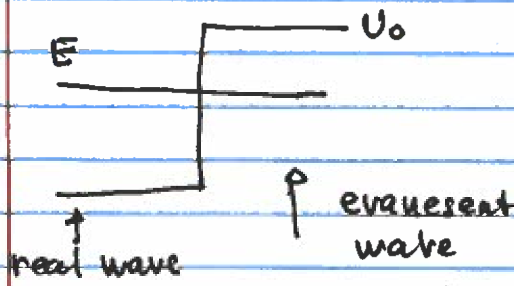
$p_0 = \sqrt{2mE}$

$\lambda = \frac{2\pi\hbar}{\sqrt{2m(E-U_0)}} > \lambda_0$

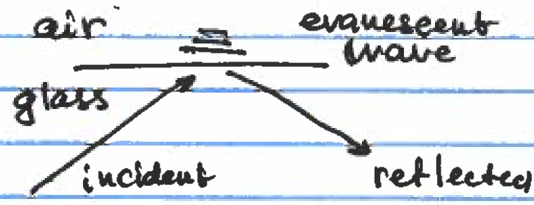
$\lambda_0/n \leftarrow$ refractive index

$\lambda_0 = \frac{2\pi\hbar}{p_0} = \frac{2\pi\hbar}{\sqrt{2mE}}$

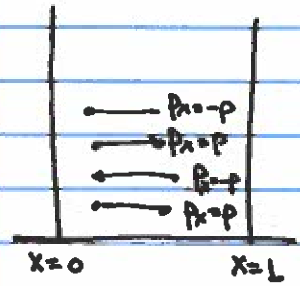
Potential step $E < U_0$



Total internal reflection



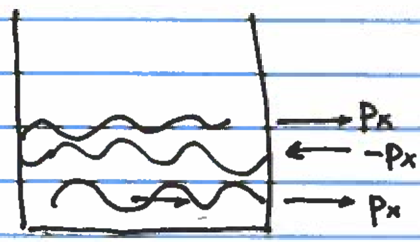
Let's get back to an infinite square well : two infinite walls



Classical particle bounces b/w two walls. Perfectly rigid walls → no energy is lost on every bounce, so particle energy / momentum value is constant, and its direction is changing.

Classical probability distribution : $P(x) = \frac{1}{L}$ constant

Quantum particles : two counter-propagating waves



Total superposition : a combination of $e^{ip_x x/\hbar}$ and $e^{-ip_x x/\hbar}$

Two "clearest" solutions : $e^{ip_x x/\hbar} + e^{-ip_x \cdot x/\hbar}$
$= 2\cos(p_x \cdot x/\hbar)$

$e^{ip_x \cdot x/\hbar} - e^{-ip_x \cdot x/\hbar} = 2i \sin(p_x \cdot x/\hbar)$

Standing waves

Actually, we just solved the corresponding Schrodinger equation

Inside: $\hat{H}\psi(x) = \frac{\hat{p}^2}{2m}\psi(x) = -\frac{\hbar^2}{2m}\psi''(x) = E\psi(x)$

$$\psi''(x) + \frac{2mE}{\hbar^2}\psi(x) = 0$$

'standing waves'

General solution form: $\psi(x) = A\cos kx + B\sin kx$

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

or $\psi(x) = c_1 e^{ikx} + c_2 e^{-ikx}$

"running waves"

Well's geometry dictates the boundary conditions: $\psi(x=0) = 0 \quad \psi(x=L) = 0$

$\psi(x=0) = A = 0$ (no cos component for a well $0 \leq x \leq L$)

$\psi(x=L) = B\sin kL = 0$

$$kL = \pi \cdot n \qquad n = 0, 1, \ldots$$

$$k_n = \frac{\pi n}{L} \qquad p_n = \hbar k = \frac{\pi\hbar n}{L} \qquad E_n = \frac{\pi^2\hbar^2 n^2}{2mL^2}$$

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{\pi n x}{L} \qquad \int_0^L |\psi_n(x)|^2 dx = 1$$

These are stationary states of a particle inside a square well $\longrightarrow$ "pure tones"

Any non-stationary state can be decompose into a set of these eigenstates

$$\psi(x) = c_1 \sin\frac{\pi x}{L} + c_2 \sin\frac{2\pi x}{L} + c_3 \frac{3\pi x}{L} + \ldots$$

(Fourier series! $\quad c_i = \int_{-\infty}^{+\infty} \psi_i^* \psi \, dx = \langle i | \psi \rangle$