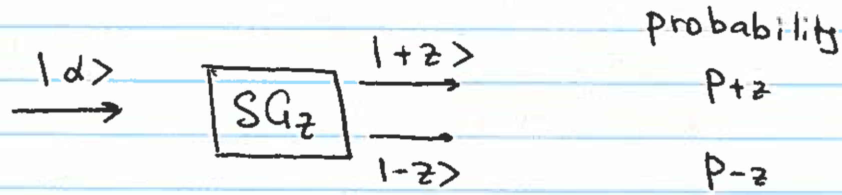
Quantum states → bra-ket notation

Quantum state: $| \{\text{state label}\} \rangle$

Example $|+z\rangle$ → the state that always emerges from positive output of SG apparatus oriented in z direction

$|-z\rangle$ → same, but from negative output



$|\pm z\rangle$ form a complete state basis

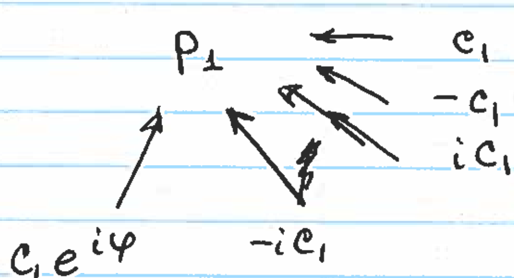
$$|\alpha\rangle = c_{+z}|+z\rangle + c_{-z}|-z\rangle$$

$$P_{+z} = |c_{+z}|^2 \qquad P_{-z} = |c_{-z}|^2$$

Probabilities are measurable values, they are real numbers $0 \leq \forall p \leq 1$

Coefficients $c_{\pm z}$ → complex numbers

Thus, even if we know / measured probabilities, we cannot get full information about the Coefficients



where φ is real

wave function is always defined up to a total phase factor, since it does not change the measurable probability $|c_1|^2 = |c_1 e^{i\varphi}|^2$

Quantum "analog" for a ~~dot~~ dot product

Bra vector $\langle\alpha|$ such that $\langle\alpha|\alpha\rangle = 1$ for any state

$$|\alpha\rangle = c_{+z}|+z\rangle + c_{-z}|-z\rangle \qquad c_{+z} = \langle +z|\alpha\rangle$$
$$\Updownarrow \qquad\qquad\qquad\qquad c_{-z} = \langle -z|\alpha\rangle$$
$$\langle\alpha| = c_{+z}^{*}\langle +z| + c_{-z}^{*}\langle -z|$$

States $|\pm z\rangle$ are orthogonal $\langle +z|-z\rangle = 0$
'Practically it means that we will never find a parcticle initially in state $|-z\rangle$ in the positive output of SG_z apparatus

Normalization check $\langle\alpha|\alpha\rangle = 1$

$$\underbrace{\left(c_{+z}^{*}\langle +z| + c_{-z}^{*}\langle -z|\right)}_{\langle\alpha|}\underbrace{\left(c_{+z}|+z\rangle + c_{-z}|-z\rangle\right)}_{|\alpha\rangle} =$$

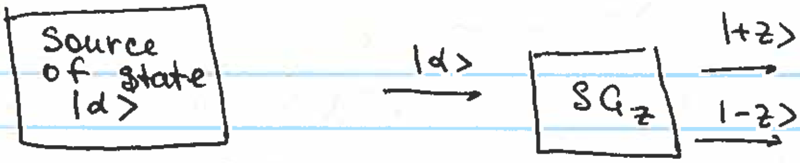
$$= |c_{+z}|^2\underbrace{\langle +z|+z\rangle}_{=1} + c_{+z}^{*}c_{-z}\underbrace{\langle +z|-z\rangle}_{=0} + c_{-z}^{*}c_{+z}\underbrace{\langle -z|+z\rangle}_{=0}$$
$$+ |c_{-z}|^2\underbrace{\langle -z|-z\rangle}_{=1} = |c_{+z}|^2 + |c_{-z}|^2 = 1$$

Let's find $|\pm x\rangle$ in $|\pm z\rangle$ basis

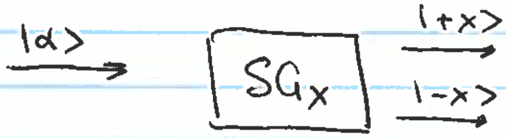
$$|+x\rangle \longrightarrow \boxed{SG_z} \begin{array}{l} \longrightarrow P_+ = 1/2 \\ \longrightarrow P_- = 1/2 \end{array}$$

$$|+x\rangle = c_+|+z\rangle + c_-|-z\rangle$$
$$|c_+|^2 = 1/2 \qquad |c_-|^2 = 1/2$$

$$c_+ = 1/\sqrt{2} \qquad c_- = 1/\sqrt{2}\, e^{i\varphi_+} \qquad |e^{i\varphi}| = 1$$



analize the output
↓
define $|\alpha\rangle$ in $|\pm z\rangle$ basis



analize the output

define $|\alpha\rangle$ in $|\pm x\rangle$ basis

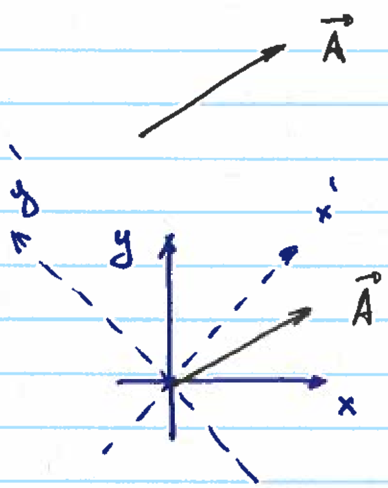
Case 1

$|\alpha\rangle = c_{+z}|+z\rangle + c_{-z}|-z\rangle \qquad |c_{+z}|^2 + |c_{-z}|^2 = 1$

Case 2

$|\alpha\rangle = c_{+x}|+x\rangle + c_{-x}|-x\rangle \qquad |c_{+x}|^2 + |c_{-x}|^2 = 1$

Vector analogy



To exist, a vector does not require a basis. But if we want to define its direction — we do!

$$\vec{A} = A_x \vec{e}_x + A_y \vec{e}_y$$

($\vec{e}_x$, $\vec{e}_y$ – unit vectors along x, y)

$$A_x = \vec{A} \cdot \vec{e}_x \quad * \qquad \vec{A} = \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

$$A_y = \vec{A} \cdot \vec{e}_y$$

$$\vec{A} = A_{x'} \vec{e}_{x'} + A_{y'} \vec{e}_{y'}$$

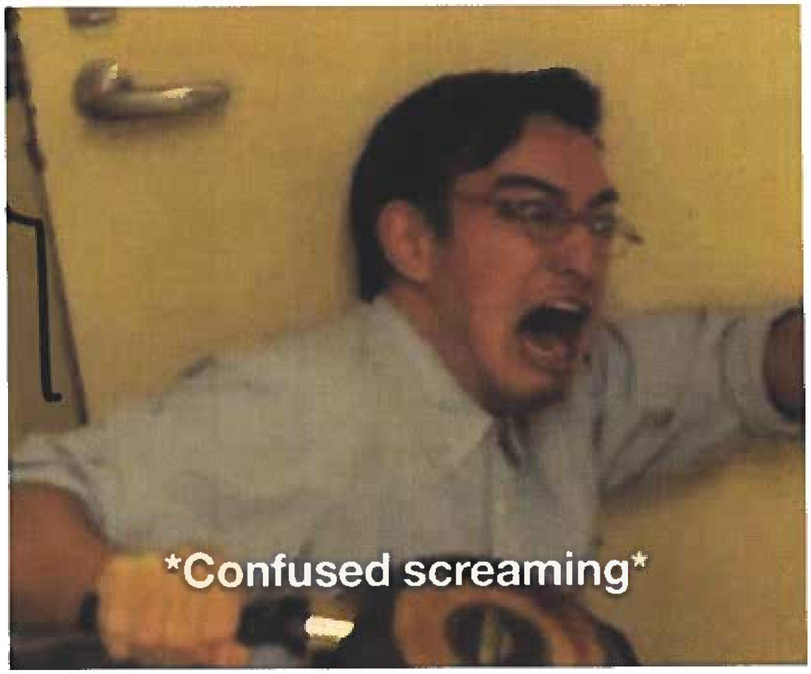
$$A_{x'} = \vec{A} \cdot \vec{e}_{x'}$$

$$A_{y'} = \vec{A} \cdot \vec{e}_{y'}$$

$$\vec{A} = \begin{pmatrix} A_{x'} \\ A_{y'} \end{pmatrix} = R \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

$$|\vec{A}| = (A_x^* \ A_y^*) \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

Physics students the first time they encounter Bra-Ket notation in quantum mechanics



$$|+x\rangle = \frac{1}{\sqrt{2}}\,|+z\rangle + \frac{1}{\sqrt{2}}\,e^{i\varphi_+}\,|-z\rangle$$

$$|-x\rangle = \frac{1}{\sqrt{2}}\,|+z\rangle + \frac{1}{\sqrt{2}}\,e^{i\varphi_-}\,|-z\rangle$$

Need extra measurement !

$$\langle +x|-x\rangle = 0$$

$$\langle +x|-x\rangle = \left(\frac{1}{\sqrt{2}}\,\langle +z| + \frac{1}{\sqrt{2}}\,e^{-i\varphi_+}\,\langle -z|\right)\left(\frac{1}{\sqrt{2}}\,|+z\rangle + \frac{1}{\sqrt{2}}\,e^{i\varphi_-}\,|-z\rangle\right)$$

$$= \frac{1}{2}\,\langle +z|+z\rangle + \cancel{\frac{1}{2}\,e^{-i\varphi_+}\,\underbrace{\langle -z|+z\rangle}_{=0}} + \cancel{\frac{1}{2}\,e^{i\varphi_-}\,\underbrace{\langle +z|-z\rangle}_{=0}}$$

$$+ \frac{1}{2}\,\langle -z|-z\rangle\,e^{i(\varphi_- - \varphi_+)} = \frac{1}{2} + \frac{1}{2}\,e^{i(\varphi_- - \varphi_+)} = 0$$

$$e^{i(\varphi_- - \varphi_+)} = -1 \qquad \varphi_- - \varphi_+ = \pi$$

For ~~[illegible]~~ x $\quad \varphi_+ = 0 \quad \varphi_- = \pi$

$$|+x\rangle = \frac{1}{\sqrt{2}}\,|+z\rangle + \frac{1}{\sqrt{2}}\,|-z\rangle$$

$$|-x\rangle = \frac{1}{\sqrt{2}}\,|+z\rangle - \frac{1}{\sqrt{2}}\,|-z\rangle$$

For y $\quad \varphi_+ = \pi/2 \quad \varphi_- = 3\pi/2$

$$|+y\rangle = \frac{1}{\sqrt{2}}\,|+z\rangle + \frac{i}{\sqrt{2}}\,|-z\rangle$$

$$|-y\rangle = \frac{1}{\sqrt{2}}\,|+z\rangle - \frac{i}{\sqrt{2}}\,|-z\rangle$$

Inverse transformations

$$|+z\rangle = \frac{1}{\sqrt{2}}\,|+x\rangle + \frac{1}{\sqrt{2}}\,|-x\rangle \qquad |+z\rangle = \frac{1}{\sqrt{2}}\,|+y\rangle + \frac{1}{\sqrt{2}}\,|-y\rangle$$

$$|-z\rangle = \frac{1}{\sqrt{2}}\,|+x\rangle - \frac{1}{\sqrt{2}}\,|-x\rangle \qquad |-z\rangle = \frac{-i}{\sqrt{2}}\,|+y\rangle + \frac{i}{\sqrt{2}}\,|-y\rangle$$

$$|\alpha\rangle = C_{+z}|+z\rangle + C_{-z}|-z\rangle = C_{+z}\left(\frac{1}{\sqrt{2}}|+x\rangle + \frac{1}{\sqrt{2}}|-x\rangle\right) +$$

$$+\ C_{-z}\left(\frac{1}{\sqrt{2}}|+x\rangle - \frac{1}{\sqrt{2}}|-x\rangle\right) = \underbrace{\left(\frac{C_{+z}+C_{-z}}{\sqrt{2}}\right)}_{C_{+x}}|+x\rangle + \underbrace{\left(\frac{C_{+z}-C_{-z}}{\sqrt{2}}\right)}_{C_{-x}}|$$

Since $|C_1|^2 + |C_2|^2 = 1$, sometimes it is convenient to use

$$|\alpha\rangle = \cos\frac{\theta}{2}|+z\rangle + e^{i\varphi}\sin\frac{\theta}{2}|-z\rangle$$

$$\left|\cos\frac{\theta}{2}\right|^2 + \left|e^{i\varphi}\sin\frac{\theta}{2}\right|^2 = 1 \quad \text{for any } \theta, \varphi$$