Brief summary

Quantum operators act on quantum states and (sometimes) change them

$$\hat{A}\,|\psi\rangle = |\varphi\rangle \quad \leftarrow \text{independent of a basis}$$

To present it in a matrix form, we have to pick a basis and write <u>everything</u> in the <u>same</u> <u>basis</u>

$$\underset{\text{vectors}}{|\psi\rangle} = \begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix} \qquad \underset{\text{vector}}{|\varphi\rangle} = \begin{pmatrix} \langle +z|\varphi\rangle \\ \langle -z|\varphi\rangle \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} \langle +z|\hat{A}|+z\rangle & \langle +z|\hat{A}|-z\rangle \\ \langle -z|\hat{A}|+z\rangle & \langle -z|\hat{A}|-z\rangle \end{pmatrix} \qquad \left(\langle +z|\hat{A}|-z\rangle\right)^* = \langle -z|\hat{A}|+z\rangle$$

But what to do if we want to switch from one basis to another? $(|\pm z\rangle \to |\pm x\rangle)$

$$|\psi\rangle = |+z\rangle\langle +z|\psi\rangle + |-z\rangle\langle -z|\psi\rangle$$

$$|+z\rangle = |+x\rangle\langle +x|+z\rangle + |-x\rangle\langle -x|+z\rangle = \overbrace{\left(|+x\rangle\langle x| + |-x\rangle\langle -x|\right)}^{\hat{I}\text{, identity operator}}|+z\rangle$$

$$|-z\rangle = |+x\rangle\langle +x|-z\rangle + |-x\rangle\langle -x|-z\rangle$$

$$|\psi\rangle = \Big[|+x\rangle\langle +x|+z\rangle + |-x\rangle\langle -x|+z\rangle\Big]\langle +z|\psi\rangle + \Big[|+x\rangle\langle +x|-z\rangle + \cancel{|-x\rangle}$$

$$+ |-x\rangle\langle -x|-z\rangle\Big]\langle -z|\psi\rangle =$$

$$= |+x\rangle\Big\{\langle +x|+z\rangle\langle +z|\psi\rangle + \langle +x|-z\rangle\langle -z|\psi\rangle\Big\} +$$

$$+ |-x\rangle\Big\{\langle -x|+z\rangle\langle +z|\psi\rangle + \langle -x|-z\rangle\langle -z|\psi\rangle\Big\}$$

In x-basis

$$|\psi\rangle = \underset{\text{vector } |\psi\rangle \text{ in x-axis}}{\begin{pmatrix} \langle +x|\psi\rangle \\ \langle -x|\psi\rangle \end{pmatrix}} = \underbrace{\begin{pmatrix} \langle +x|+z\rangle & \langle +x|-z\rangle \\ \langle -x|+z\rangle & \langle -x|-z\rangle \end{pmatrix}}_{\text{Transformation matrix}} \underset{\text{Vector } |\psi\rangle \text{ in z-basis}}{\begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}}$$

Notice that I used $\{|z\rangle\}$ and $\{|x\rangle\}$ basis ~~as exam~~ transformation as an example, but without any specific details, so these can be any two bases.

$$|+z\rangle = \frac{1}{\sqrt{2}}\left(|+x\rangle + |-x\rangle\right) \qquad |-z\rangle = \frac{1}{\sqrt{2}}\left(|+x\rangle - |-x\rangle\right)$$

$$\mathcal{J}_{z\to x} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Inverse transformation

$$\mathcal{J}_{x\to z} = \begin{pmatrix} \langle +z|+x\rangle & \langle +z|-x\rangle \\ \langle +z|+x\rangle & \langle -z|-x\rangle \end{pmatrix}$$

$$|+x\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle + |-z\rangle\right) \qquad |-x\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle - |-z\rangle\right)$$

$$\mathcal{J}_{x\to z} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

In general

$$\mathcal{J}_{\text{basis } 1\to \text{basis } 2} = \mathcal{J}^{\dagger}_{\text{basis } 2\to\text{basis } 1}$$

Hermitian conjugate

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad A^{\dagger} = (A^{*})^{T} = \begin{pmatrix} a_{11}^{*} & a_{21}^{*} \\ a_{12}^{*} & a_{22}^{*} \end{pmatrix}$$

What about an operator?

For example, let assume we know matrix representation of $\hat{A}$ in z-basis

$$\hat{A} = \begin{pmatrix} A_{11}^z & A_{12}^z \\ A_{21}^z & A_{22}^z \end{pmatrix}$$

but ~~I~~ we need to calculate its action on a state in ~~a~~ the $|x\rangle$ basis?

$$|\psi\rangle = \begin{pmatrix} \cancel{c_1^x} \\ \cancel{c_2^x} \end{pmatrix} = \begin{pmatrix} \langle +x|\psi\rangle \\ \langle -x|\psi\rangle \end{pmatrix}$$

$$|\varphi\rangle = \hat{A}\,|\psi\rangle = \begin{pmatrix} A_{11}^z & A_{12}^z \\ A_{21}^z & A_{22}^z \end{pmatrix} \begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix} = \cancel{\begin{pmatrix} A_{11}^z & A_{12}^z \\ A_{21}^z & A_{22}^z \end{pmatrix}}$$

$$= \begin{pmatrix} A_{11}^z & A_{12}^z \\ A_{21}^z & A_{22}^z \end{pmatrix} \mathcal{J}_{x\to z} \begin{pmatrix} \langle +x|\psi\rangle \\ \langle -x|\psi\rangle \end{pmatrix} = \begin{pmatrix} \langle \varphi|+z\rangle \\ \langle \varphi|-z\rangle \end{pmatrix}$$

we are not done, since the answer will be in z- ~~re~~ basis, and we want it in x-basis:

$$\begin{pmatrix} \langle +x|\varphi\rangle \\ \langle -x|\varphi\rangle \end{pmatrix} = \mathcal{J}_{z\to x} \begin{pmatrix} \langle +z|\varphi\rangle \\ \langle -z|\varphi\rangle \end{pmatrix}$$

So, the total expression is

$$\begin{pmatrix} \langle +x|\varphi\rangle \\ \langle -x|\varphi\rangle \end{pmatrix} = \underbrace{\mathcal{J}_{z\to x} \begin{pmatrix} A_{11}^z & A_{12}^z \\ A_{21}^z & A_{22}^z \end{pmatrix} \mathcal{J}_{x\to z}}_{\text{This matrix product is the matrix representation of } \hat{A} \text{ in x-basis}} \begin{pmatrix} \langle +x|\psi\rangle \\ \langle -x|\psi\rangle \end{pmatrix}$$

Thus, the operator matrix transformation eqn

$$\begin{pmatrix} \langle +x|\hat{A}|+x\rangle & \langle +x|\hat{A}|-x\rangle \\ \langle -x|\hat{A}|+x\rangle & \langle -x|\hat{A}|+x\rangle \end{pmatrix} = \mathcal{T}_{z\to x} \begin{pmatrix} \langle +z|\hat{A}|+z\rangle & \langle +z|\hat{A}|-z\rangle \\ \langle -z|\hat{A}|+z\rangle & \langle -z|\hat{A}|-z\rangle \end{pmatrix} \mathcal{T}_{x\to z}$$

<u>Example 1</u> $\quad \hat{J}_x$ in z-basis

$\hat{J}_x = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in x-basis

& $\mathcal{T}_{z\to x} = \mathcal{T}_{x\to z} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\hat{J}_x\Big|_{z\text{-basis}} = \mathcal{T}_{z\to x}\, J_x\Big|_{x\text{-basis}}\, \mathcal{T}_{x\to z} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{\hbar}{2}\,\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & +1 \\ -1 & 1 \end{pmatrix} = \frac{\hbar}{2}\,\frac{1}{2}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{J}_x\Big|_{z\text{-basis}} = \frac{\hbar}{2}\hat{\sigma}_x \qquad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Example 2 $\quad \hat{J}_y$ in z-basis

$\hat{J}_y = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in y-basis

$$\mathcal{T}_{y\to z} = \begin{pmatrix} \langle +z|+y\rangle & \langle +z|-y\rangle \\ \langle -z|+y\rangle & \langle -z|-y\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

since $|\pm y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle \pm i|-z\rangle)$ $\qquad |+z\rangle = \frac{1}{\sqrt{2}}(|+y\rangle + |-y\rangle)$

$\qquad |-z\rangle = -\frac{1}{\sqrt{2}}(|+y\rangle - |-y\rangle)$

$$\hat{\mathcal{T}}_{z\to y} = (\hat{\mathcal{T}}_{y\to z})^{\dagger} = \left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\right)^{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$\hat{J}_y\Big|_{z\text{-basis}} = \hat{\mathcal{T}}_{y\to z}\, \hat{J}_y\Big|_{y\text{-basis}}\, \hat{\mathcal{T}}_{z\to y} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} =$$

$$= \frac{\hbar}{2}\,\frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix} = \frac{\hbar}{2}\,\frac{1}{2}\begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{J}_y\Big|_{z\text{-basis}} = \frac{\hbar}{2}\hat{\sigma}_y \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Pauli matrices

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Universally used to describe spin orientation (both classically and quantum-ly), light polarization, any two-level quantum system.