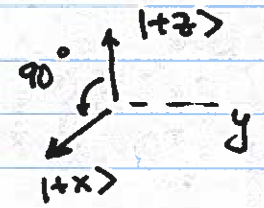
Basis transformation

We can transform the state vectors using the rotation operator



$$|+x\rangle = R\left(\frac{\pi}{2}\vec{j}\right)|+z\rangle \qquad \langle +x| = \langle +z| R^{\dagger}\left(\frac{\pi}{2}\vec{j}\right)$$

$$|-x\rangle = R\left(\frac{\pi}{2}\vec{j}\right)|-z\rangle \qquad \langle -x| = \langle -z| R^{\dagger}\left(\frac{\pi}{2}\vec{j}\right)$$

$$|d\rangle = c_+|+z\rangle + c_-|-z\rangle \quad \text{or} \quad |d\rangle = \tilde{c}_+|+x\rangle + \tilde{c}_-|-x\rangle$$

$|+z\rangle\langle +z| + |-z\rangle\langle -z|$

$$\begin{pmatrix} \tilde{c}_+ \\ \tilde{c}_- \end{pmatrix} = \begin{pmatrix} \langle +x|d\rangle \\ \langle -x|d\rangle \end{pmatrix} = \begin{pmatrix} \langle +x|+z\rangle\langle +z|d\rangle + \langle +x|-z\rangle\langle -z|d\rangle \\ \langle -x|+z\rangle\langle +z|d\rangle + \langle -x|-z\rangle\langle -z|d\rangle \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \langle +x|+z\rangle & \langle +x|-z\rangle \\ \langle -x|+z\rangle & \langle -x|-z\rangle \end{pmatrix}}_{\text{basis transformation}} \begin{pmatrix} \langle +z|d\rangle \\ \langle -z|d\rangle \end{pmatrix} = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

basis

$$\hat{\tau}_{z\to x}$$

$$\hat{\tau}_{z\to x} = \begin{pmatrix} \langle +x|+z\rangle & \langle +x|-z\rangle \\ \langle -x|+z\rangle & \langle -x|-z\rangle \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$\hat{\tau}_{x\to z} = \begin{pmatrix} \langle +z|+x\rangle & \langle +z|-x\rangle \\ \langle -z|+x\rangle & \langle -z|-x\rangle \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$\hat{\tau}_{z\to y} = \begin{pmatrix} \langle +y|+z\rangle & \langle +y|-z\rangle \\ \langle -y|+z\rangle & \langle -y|-z\rangle \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}$$

$$\hat{\tau}_{y\to z} = \begin{pmatrix} \langle +z|+y\rangle & \langle +z|-y\rangle \\ \langle -z|+y\rangle & \langle -z|-y\rangle \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & +i/\sqrt{2} \end{pmatrix}$$

What if we know the operator matrix in one basis, but our states are defined in a different basis?

For example: since $|\pm x\rangle$ are the eigenstates of $\hat{J}_x$, in $|\pm x\rangle$ basis the matrix for $\hat{J}_x$ is $\begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}$

$|fin\rangle = \hat{J}_x |ini\rangle \longleftarrow$ valid notation in general

But in case we want to use matrix form, all states / operators must be in the same state so

$$\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \hat{J}_x\Big|_{\text{in } z\text{-basis}} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

or

$$\begin{pmatrix} \tilde{f}_+ \\ \tilde{f}_- \end{pmatrix} = \hat{J}_x\Big|_{\text{in } x\text{-basis}} \begin{pmatrix} \tilde{c}_+ \\ \tilde{c}_- \end{pmatrix} \neq \cancel{\hat{J}_x\Big|_{\text{in } z\text{-basis}}}$$

$$= \hat{J}_x\Big|_{\text{in } x\text{-bas}} \hat{\tau}_{z\to x} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

$$\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \hat{\tau}_{x\to z} \begin{pmatrix} \tilde{f}_+ \\ \tilde{f}_- \end{pmatrix} = \hat{\tau}_{x\to z} \hat{J}_x\Big|_{\text{in } x\text{-bas}} \hat{\tau}_{z\to x} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

$$\hat{J}_x\Big|_{\text{in } z\text{-basis}} = \hat{\tau}_{x\to z} \hat{J}_x\Big|_{\text{in } x\text{-basis}} \hat{\tau}_{z\to x}$$

$$\hat{J}_x\Big|_{\text{in } z\text{-basis}} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} =$$

$$= \frac{\hbar}{2} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2} \hat{\sigma}_x$$