Physics 313 Midterm test #2
November 9, 2022

Name (please print): Solutions

This test is administered under the rules and regulations of the honor system of the College of William & Mary.

Signature: ______________________________

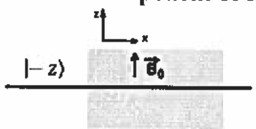
Final score: ___________

- *Show all work to receive credit, and circle your final answers. This exam is closed book, but you can use a prepared index card with reference information that you have prepared.*

Problem 1 (40 points)

Note: The magnetic moment for an electron is $\vec{\mu} = \frac{-eg}{2mc}$, and the Larmor frequency is $\omega_0 = \frac{egB_0}{2mc}$.

(a) A spin 1/2 particle passes through a Stern-Gerlach apparatus that only transmits particles with $S_z = -\hbar/2$ and enters the region of a uniform magnetic field pointing in the direction $\vec{B} = B_0\vec{k}$. What is the energy of the particle? If the z component of its spin is measured, what is its value as a function of time?



(b) The Stern-Gerlach apparatus is now rotated to transmit particles in the state $|+x\rangle$. The magnetic field is still in the z direction $\vec{B} = B_0\vec{k}$. If it takes particle time T to traverse the magnetic field-filled region, what is the probability of detecting it in state with $S_y = \hbar/2$?

(c) Finally, the magnetic field direction changes to be in the $x - z$ plane $\vec{B} = \frac{1}{\sqrt{2}}(B_0\vec{i} + B_0\vec{k})$. Write down the Hamiltonian of a particle in this magnetic field, and calculate the average energy for a particle in the state $|-z\rangle$.

a) $\hat{H} = -\hat{\vec{\mu}}\vec{B} = \frac{egB_0}{2mc}\hat{S}_z = \omega_0 \hat{S}_z$

$\langle E\rangle = \langle -z|\hat{H}|-z\rangle = \omega_0\langle -z|\hat{S}_z|-z\rangle = -\frac{\hbar\omega_0}{2}$

$\hat{H}|-z\rangle = -\frac{\hbar\omega_0}{2}|-z\rangle$ eigenstate of $\hat{H}$, a stationary state,

so it always gives $-\hbar/2$ outcome for S_z measurement

b) $|+x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + |-z\rangle) = |\psi(x,0)\rangle$

$|\psi(x,t)\rangle = e^{-i\hat{H}t/\hbar}|+x\rangle = \frac{1}{\sqrt{2}}\{e^{-i\omega_0\hat{S}_z/\hbar}(|+z\rangle + |-z\rangle) =$

$= \frac{1}{\sqrt{2}}\left(|+z\rangle e^{-i\omega_0 t/2} + |-z\rangle e^{+i\omega_0 t/2}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega_0 t/2} \\ e^{i\omega_0 t/2} \end{pmatrix}$

$P_{+y} = |\langle +y|\psi(x,t)\rangle|^2 = \frac{1}{4}\left|(1\ -i)\begin{pmatrix} e^{-i\omega_0 t/2} \\ e^{i\omega_0 t/2} \end{pmatrix}\right|^2 = \frac{1}{4}\left|e^{-i\omega_0 t/2} - i e^{i\omega_0 t/2}\right|^2$

$= \frac{1+\sin\omega_0 t}{2}$

c) $\hat{H} = -\hat{\vec{\mu}}\vec{B} = \frac{1}{\sqrt{2}}\omega_0\hat{S}_x + \frac{1}{\sqrt{2}}\omega_0\hat{S}_z = \frac{\omega_0\hbar}{2}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

$\langle E\rangle = \langle -z|\hat{H}|-z\rangle = \frac{\omega_0\hbar}{2}(0, 1)\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\omega_0\hbar}{2\sqrt{2}}$

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Problem 2 (40 points)

A particle of mass m in a potential well $V(x) = \begin{cases} 0 & 0 \leq x \leq L \\ \infty & \text{elsewhere} \end{cases}$ is in the initial state $\psi(x, t=0) = \psi(x)$.

(a) Prove that $E_n = \frac{\pi^2 \hbar^2 n^2}{2mL^2}$ are the energy values corresponding to the Hamiltonian eigenstates $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right)$.

For questions (b,c) assume that the initial state of the particle is

$$\psi(x) = \begin{cases} \frac{i}{2}\sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) + \frac{1}{2}\sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right) - \frac{1}{\sqrt{2}}\sqrt{\frac{2}{L}} \sin\left(\frac{3\pi x}{L}\right) & 0 \leq x \leq L \\ 0 & \text{elsewhere} \end{cases}$$

$$\boxed{\psi_n(x,t) = \psi_n(x)\, e^{-iE_n t/\hbar}}$$

(b) Write the time evolution of this state $\psi(x, t)$.

(c) What is the average energy $\langle E \rangle$ of the particle in this state?

(d) Now at $t = 0$ we are going to place the particle initially exactly in the center of the well, and approximate its initial wave function as a delta function: $\psi(x) = \delta(x - L/2)$. What is the probability of measuring the particle in the state n (with the energy E_n) for $t > 0$?

a) $\hat{H} = \frac{\hat{P}^2}{2m} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$

$$\hat{H}\psi_n(x) = -\frac{\hbar^2}{2m}\sqrt{\frac{2}{L}}\left(-\frac{\pi^2 n^2}{L^2}\sin\frac{\pi n x}{L}\right) =$$

$$= \frac{\pi^2 n^2 \hbar^2}{2mL^2}\sqrt{\frac{2}{L}}\sin\frac{\pi n x}{L} = E_n \psi_n(x)$$

b) $$\psi(x,t) = \frac{i}{2}\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}\, e^{-i\frac{\pi^2 \hbar t}{2mL^2}} + \frac{1}{2}\sqrt{\frac{2}{L}}\left(\frac{2\pi x}{L}\right) e^{-i\frac{2\pi^2 \hbar t}{mL^2}} - \frac{1}{\sqrt{2}}\sqrt{\frac{2}{L}}\sin\frac{3\pi x}{L}\, e^{-\frac{9\pi^2 \hbar}{2mL}}$$

c) $$\langle E \rangle = \left|\frac{i}{2}\right|^2 \cdot \frac{\pi^2\hbar^2}{2mL^2} + \left|\frac{1}{2}\right|^2 \frac{2\pi^2\hbar^2}{mL^2} + \left|-\frac{1}{\sqrt{2}}\right|^2 \frac{9\pi^2\hbar^2}{2mL^2} = \frac{1}{4}E_1 + \frac{1}{4}\cdot 4E_1 + \frac{1}{2}\cdot 9E_1 =$$

$$= \frac{23}{4}E_1 = \frac{23\pi^2\hbar^2}{8mL^2}$$

d) $$P_n = \left|\int_0^L \psi_n^*(x)\psi(x)\,dx\right|^2 = \left|\sqrt{\frac{2}{L}}\int_0^L \sin\frac{\pi n x}{L}\,\delta\left(x - \frac{L}{2}\right)dx\right|^2 = \left|\sqrt{\frac{2}{L}}\sin\frac{\pi n}{2}\right|^2$$

$P_n = 0$ if n is even

$P_n = \frac{2}{L}$ if n is odd

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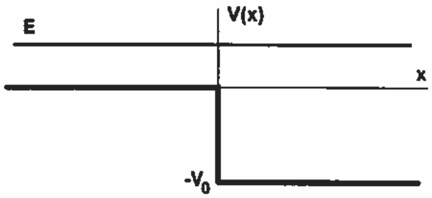
Problem 3 (20 points)

Calculate the reflection probability for a particle of mass m moving with the energy $E > 0$ from negative infinity in $+x$ direction when it encounter a potential step $V(x) = \begin{cases} 0 & x \le 0 \\ -V_0 & x > 0 \end{cases}$ shown below.



$x<0$ $\quad \dfrac{p^2}{2m} = E$

$p = \sqrt{2mE}$

$x>0$ $\quad \dfrac{p_1^2}{2m} - V_0 = E$

$p_1 = \sqrt{2m(E+V_0)}$

$$\psi(x) = \begin{cases} Ae^{ipx/\hbar} + Be^{-ipx/\hbar} & x<0 \\ Ce^{ip_1x/\hbar} & x>0 \end{cases}$$

$x=0$: continuity $\quad A+B = C$

smoothness $\quad \dfrac{ip}{\hbar}(A-B) = \dfrac{ip_1}{\hbar}C$

$$p(A-B) = p_1(A+B)$$

$$B = \frac{p-p_1}{p+p_1}A$$

Reflection probability

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{p-p_1}{p+p_1}\right)^2$$

Often $k = p/\hbar$ and $k_1 = p_1/\hbar$ notation is used, then

$$R = \left(\frac{k-k_1}{k+k_1}\right)^2$$

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