# Physics 313 Midterm test #1
# October 9, 2024

Name (please print): solutions

*This test is administered under the rules and regulations of the honor system of the College of William & Mary.*

Signature: ______________________

Final score: ___________

*Show all work to receive credit, and circle your final answers. This exam is closed book, but you can use a prepared index card with reference information that you have prepared.*

**Problem 1**(30 points)

A spin-1 particle is prepared in the quantum state $|\alpha\rangle = \frac{3}{4}|1,1\rangle + \frac{\sqrt{2}+i}{4}|1,0\rangle - \frac{1}{2}|1,-1\rangle$.

(a) If a spin-1 particle is sent through a Stern-Gerlach apparatus that separates particles according to their $S_z$ spin component, what are the possible values of the $S_z$ spin component? If the particle is in the state $|\alpha\rangle$, What are the probabilities of each outcome?


$P_+ = |\langle 1,1|\alpha\rangle|^2 = 9/16$

$P_0 = |\langle 1,0|\alpha\rangle|^2 = \left|\frac{\sqrt{2}+i}{4}\right|^2 = \frac{3}{16}$

$P_1 = |\langle 1,-1|\alpha\rangle|^2 = 1/4$

(b) Now we have rotated the SG apparatus by 90° such that it measures $S_x$ spin component, as shown. The eigenstates of the $\hat{S}_x$ in $z$-basis are:

$|+x\rangle = \frac{1}{2}\left(|1,1\rangle + \sqrt{2}|1,0\rangle + |1,-1\rangle\right)$,

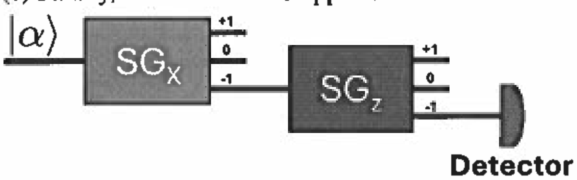
$|0_x\rangle = \frac{1}{\sqrt{2}}\left(|1,1\rangle - |1,-1\rangle\right)$, and

$|-x\rangle = \frac{1}{2}\left(|1,1\rangle - \sqrt{2}|1,0\rangle + |1,-1\rangle\right)$.

What is the probability that a particle will be measured in the state $|+x\rangle$?


$P_{|+x\rangle} = |\langle +x|\alpha\rangle|^2 = \left|\frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}\begin{pmatrix}3/4 \\ \sqrt{2}+i/4 \\ -1/2\end{pmatrix}\right|^2 = \frac{1}{4}\left|\frac{3}{4} + \frac{\sqrt{2}i}{4}\right|^2 = \frac{11}{64}$

(c) Finally, we stack two SG apparati as shown. What is the probability that the particle will reach the detector?




Probability to get through SG x

$P_1 = |\langle -x|\alpha\rangle|^2 = \left|\frac{1}{2}\begin{pmatrix}1 & -\sqrt{2} & 1\end{pmatrix}\begin{pmatrix}3/4 \\ \sqrt{2}+i/4 \\ -1/2\end{pmatrix}\right|^2 =$

$= \frac{1}{4}\left|-\frac{1}{4} - \frac{\sqrt{2}i}{4}\right|^2 = \frac{3}{64}$

Probability to get through SG z

$P_2 = |\langle 1,-1|-x\rangle|^2 = \left|\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}1/2 \\ -1/\sqrt{2} \\ 1/2\end{pmatrix}\right|^2 = \frac{1}{4}$

Total probability: $P_{tot} = P_1 \cdot P_2 = \frac{3}{256}$

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**Problem 2** (35 points)

The operator $\hat{\Sigma}$, acting on a spin-1/2 particle, in the $z$-basis is described by the following matrix:

$$\hat{\Sigma} = \begin{pmatrix} 1/2 & \sqrt{3}i/2 \\ -\sqrt{3}i/2 & -1/2 \end{pmatrix}$$

(a) Someone calculated the eigenvalues of this operator to be $\lambda_\pm = \pm 1$. Find the corresponding eigenstates $|\pm\rangle$ of this operator in the $z$-basis.

(b) What is the average value of this operator in the state $|+x\rangle$?

(c) What is the probability that when $\Sigma$ is measured for a particle in the state $|-x\rangle$, it will yield the value $\lambda_+ = 1$?

(d) A particle originally in the state $|+y\rangle$ is acted upon first the operator $\Sigma$, and them by the operator $S_z$. Calculate the final state of the particle.

a) $\hat{F}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda_\pm \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ and $|c_1|^2 + |c_2|^2 = 1$

$\lambda = 1$

$$\begin{pmatrix} 1/2 & \sqrt{3}i/2 \\ -\sqrt{3}i/2 & -1/2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\frac{1}{2}c_1 + \frac{\sqrt{3}i}{2}c_2 = c_1$$

$$\frac{1}{2}c_1 = \frac{\sqrt{3}i}{2}c_2 \Rightarrow c_2 = \frac{c_1}{\sqrt{3}i}$$

Normalization:

$$|c_1|^2 + |c_2|^2 = |c_1|^2\left(1 + \frac{1}{3}\right) = 1$$

$$|c_1|^2 = \frac{3}{4} \qquad c_1 = \sqrt{3}/2$$

$$\Downarrow$$

$$c_2 = -\frac{i}{2}$$

$$|+\rangle = \begin{pmatrix} \sqrt{3}/2 \\ -i/2 \end{pmatrix}$$

$\lambda = -1$

$$\begin{pmatrix} 1/2 & \sqrt{3}i/2 \\ -\sqrt{3}i/2 & -1/2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\frac{1}{2}c_1 + \frac{\sqrt{3}i}{2}c_2 = -c_1$$

$$\frac{3}{2}c_1 = -\frac{\sqrt{3}i}{2}c_2 \qquad c_2 = \sqrt{3}ic_1$$

Normalization

$$|c_1|^2 + |c_2|^2 = 1$$

$$|c_1|^2 + 3|c_1|^2 = 1 \qquad c_1 = \frac{1}{2}$$

$$\Downarrow$$

$$c_2 = \frac{\sqrt{3}i}{2}$$

$$|-\rangle = \begin{pmatrix} 1/2 \\ \sqrt{3}i/2 \end{pmatrix}$$

b) $$\langle\hat{\Sigma}\rangle = \frac{1}{\sqrt{2}}(1 \;\; 1)\begin{pmatrix} 1/2 & \sqrt{3}i/2 \\ -\frac{\sqrt{3}i}{2} & -1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cdot\frac{1}{\sqrt{2}} = \frac{1}{2}(1 \;\; 1)\begin{pmatrix} 1/2 + \sqrt{3}i/2 \\ -1/2 - \sqrt{3}i/2 \end{pmatrix} = 0$$

c) $$P_{|+\rangle} = |\langle + |\cancel{x}\rangle|^2 = \left|\left(\frac{\sqrt{3}}{2} \;\; \frac{i}{2}\right)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}\right|^2 = \frac{1}{2}\left|\frac{\sqrt{3}}{2} - \frac{i}{2}\right|^2 = \frac{1}{2}$$

d) $$|\text{final}\rangle = \hat{S}_z\hat{\Sigma}|+y\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1/2 & \sqrt{3}i/2 \\ -\sqrt{3}i/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1/2 & \sqrt{3}i/2 \\ \sqrt{3}i/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} =$$

$$= \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} (1-\sqrt{3})/2 \\ i(1+\sqrt{3})/2 \end{pmatrix} = \frac{\hbar}{4\sqrt{2}}\begin{pmatrix} 1-\sqrt{3} \\ i(1+\sqrt{3}) \end{pmatrix}$$

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**Problem 3** (35 points)
While discussing quantum computing, we came across the Hadamar operator $\hat{\mathcal{H}}$ that is described by the following matrix in the $z$-basis:

$$\hat{\mathcal{H}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(a) Calculate their commutator $\hat{C} = [\hat{\mathcal{H}}, \hat{S}_z]$.
(b) Calculate the uncertainties $\Delta\mathcal{H}$ and $\Delta\hat{S}_z$ for the $|+y\rangle$ eigenstate of $\hat{S}_y$. Check to see if the uncertainty relation $\Delta\mathcal{H}\Delta\hat{S}_z \geq \frac{1}{2}|\langle\hat{C}\rangle|$ is valid.

*Reminder: the uncertainty of an operator is defined as* $\Delta A = \sqrt{\langle(\hat{A} - \langle\hat{A}\rangle)^2\rangle}$.

a) $\hat{C} = [\hat{H}, \hat{S}_z] = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =$

$= \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} - \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b) $\langle\hat{H}\rangle = \langle +y|\hat{H}|+y\rangle = \frac{1}{2\sqrt{2}}(1 \ -i)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = 0$

$\hat{H}^2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{\mathbb{1}} \quad \langle\hat{H}^2\rangle = \langle +y|\hat{\mathbb{1}}|+y\rangle = 1$

$\Delta H = \sqrt{\langle H^2\rangle - \langle H\rangle^2} = 1$

$\langle\hat{S}_z\rangle = \langle +y|\hat{S}_z|+y\rangle = 0$

$\hat{S}_z^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4}\hat{\mathbb{1}} \quad \langle S_z^2\rangle = \frac{\hbar^2}{4}$

$\Delta S_z = \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} = \frac{\hbar}{2}$

$\langle\hat{C}\rangle = \frac{\hbar}{2\sqrt{2}}(1 \ -i)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{2\sqrt{2}}(1 \ -i)\begin{pmatrix} -i \\ 1 \end{pmatrix} = -\frac{i\hbar}{\sqrt{2}}$

$\Delta H \cdot \Delta S = \frac{\hbar}{2} \geq \frac{1}{2}|\langle\hat{C}\rangle|^2 = \frac{\hbar}{2\sqrt{2}}$ valid

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