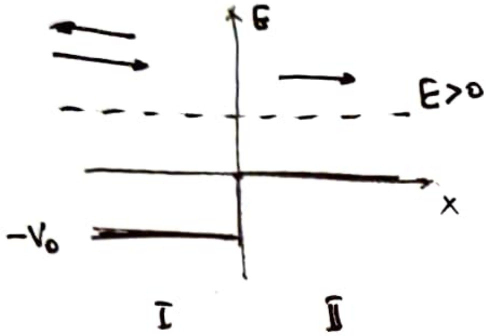
Problem 1 (15 points)

Consider a particle with mass m and energy $E > 0$ that experiences a one-dimensional potential given by $V(x) = \begin{cases} 0 & x \geq 0 \\ -V_0 & x < 0 \end{cases}$
If the particle is incident from the left (i.e. from $x = -\infty$), what is the value of V_0 such that there is a 25% probability that the particle will "reflect" back?



$$\psi(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & x<0 \\ Ce^{ik_2x} & x>0 \end{cases}$$

$$k_1 = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

$$k_2 = \sqrt{\frac{2mE}{\hbar^2}}$$

Boundary conditions

$x=0$ $\quad A+B=C$

$$ik_1(A-B) = ik_2C = ik_2(A+B)$$

$$k_1(A-B) = k_2(A+B)$$

$$(k_1-k_2)A = (k_1+k_2)B \qquad B = A\left(\frac{k_1-k_2}{k_1+k_2}\right)$$

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2 = \frac{1}{4}$$

$$k_1-k_2 = \frac{1}{2}(k_1+k_2)$$

$$k_1 = 3k_2$$

$$\sqrt{\frac{2m(E+V_0)}{\hbar^2}} = 3\sqrt{\frac{2mE}{\hbar^2}}$$

$$E+V_0 = 9E$$

$$V_0 = 8E$$

Show all work to receive credit, and circle your final answers. This exam is closed book, but you can use the index card with reference information that you have prepared.

Problem 2 (20 points)

A quantum state of an electron spin in z bases at $t=0$ can be written as: $|\alpha\rangle = A\begin{pmatrix} 1-i \\ \sqrt{3} \end{pmatrix}$

(a) Determine A by normalizing this state.
(b) If you measured S_z of this electron, what possible values could you get, and with what probabilities?
(c) If the y component of spin is measured, what is the probability to get the value $-\hbar/2$?
(d) What is the expectation value of $\hat{S}_x + \hat{S}_y$?

(e) As a part of the quantum circuit, the electron is sent through the phase gates $\hat{\Phi} = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & 1 \end{pmatrix}$, followed by the modified Hadamar gate $\hat{H}_m = \begin{pmatrix} \sqrt{2/5} & \sqrt{3/5} \\ \sqrt{3/5} & -\sqrt{2/5} \end{pmatrix}$. Calculate the output state.

4pnt a) $|A|^2 \left(\underbrace{|1-i|^2}_{2} + 3 \right) = 5|A|^2 = 1$ $\quad A = \sqrt{1/5}$

4pnt b) Possible values $\quad S_z = +\hbar/2 \quad P_{+z} = \frac{|1-i|^2}{5} = \frac{2}{5}$

$S_z = -\hbar/2 \quad P_{-z} = 3/5$

4pnt c) $P_{-y} = |\langle -y|\alpha\rangle|^2 = \left| \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{5}} (1 \ \ i) \begin{pmatrix} 1-i \\ \sqrt{3} \end{pmatrix} \right|^2 = \frac{1}{10} |(1-i+i\sqrt{3})|^2 =$

$= \frac{1}{10} |1+i(\sqrt{3}-1)|^2 = \frac{1}{10} \left(1 + \underbrace{|\sqrt{3}-1|^2}_{4-2\sqrt{3}}\right) = \frac{5-2\sqrt{3}}{10}$

4pnt d) $\hat{S}_x + \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}$

$\langle S_x + S_y \rangle = \langle \alpha | \hat{S}_x + \hat{S}_y | \alpha \rangle = \frac{\hbar}{2} \cdot \frac{1}{5} \left((1+i \ \ \sqrt{3}) \underbrace{\begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix} \begin{pmatrix} 1-i \\ \sqrt{3} \end{pmatrix}}_{\begin{pmatrix} \sqrt{3}(1-i) \\ 2 \end{pmatrix}} \right)$

$= \frac{\hbar}{10} \left[\underbrace{(1+i) \cdot \sqrt{3}(1-i)}_{2\sqrt{3}} + 2\sqrt{3} \right] = \frac{4\sqrt{3}}{10} \hbar = \frac{2\sqrt{3}}{5} \hbar$

4pnt e) $|out\rangle = \hat{H}_m \hat{\Phi} |\alpha\rangle = \begin{pmatrix} \sqrt{2/5} & \sqrt{3/5} \\ \sqrt{3/5} & -\sqrt{2/5} \end{pmatrix} \begin{pmatrix} \frac{1+i}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-i/\sqrt{5} \\ \sqrt{3/5} \end{pmatrix} =$

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$= \begin{pmatrix} \sqrt{2/5} & \sqrt{3/5} \\ \sqrt{3/5} & -\sqrt{2/5} \end{pmatrix} \begin{pmatrix} 2/\sqrt{10} \\ \sqrt{3/5} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Problem 3 (20 points)

A hydrogen atom at time $t = 0$ is found in the state $|\alpha\rangle = \frac{1}{3}|300\rangle - i\frac{2\sqrt{2}}{3}|421\rangle$ (here we use the nomenclature $|nlm\rangle$ for a quantum state with quantum numbers n, l, m.

(a) Is this a stationary state? Explain your answer.

(b) What possible values of E, L^2 and L_z could arise from a measurement of this atom?

(c) What are the expectation values for measurements of each of E, L^2 and L_z?

(d) Write the time-evolved expression for this state $|\alpha(t)\rangle$.

(e) At what time this state becomes orthogonal to $|\beta\rangle = \frac{1}{3}|300\rangle + \frac{2\sqrt{2}}{3}|421\rangle$. You can assume that $\{|nlm\rangle\}$ basis is orthonormal.

2pts a) Not a stationary state since $\hat{H}|\alpha\rangle \neq E|\alpha\rangle$

~~that~~ $|\alpha\rangle$ is not the eigentstate of the Hamiltonian

6pts b) E: $n=3$: $E_3 = -\frac{E_R}{9}$ $n=4$: $E_4 = -\frac{E_R}{16}$; $\hat{L}^2; \hat{L}_z$: $l=0, m=0$: $0, 0$ $(P = 1/9)$ or $l=2, m=1$: $6\hbar^2$ & $\hbar$ $(P = 8/9)$

$P_3 = 1/9$ $P_4 = 8/9$

6pts c) $\langle E\rangle = -\frac{E_R}{9}\cdot\frac{1}{9} + -\frac{E_R}{16}\cdot\frac{8}{9} = -\frac{11}{162}E_R$

$\langle L^2\rangle = 0\cdot\frac{1}{9} + 6\hbar^2\cdot\frac{8}{9} = \frac{16}{3}\hbar^2$

$\langle L_z\rangle = 0\cdot\frac{1}{9} + \hbar\cdot\frac{8}{9} = \frac{8}{9}\hbar$

4pnts d) $|\alpha(t)\rangle = \frac{1}{3}e^{+iE_Rt/9\hbar}|300\rangle - i\frac{2\sqrt{2}}{3}e^{+iE_Rt/16\hbar}|421\rangle$

2pts e) $\langle\beta|\alpha(t)\rangle = \left(\frac{1}{3}\langle 300| + \frac{2\sqrt{2}}{3}\langle 421|\right)\left(\frac{1}{3}e^{+iE_Rt/9\hbar}|300\rangle + i\frac{2\sqrt{2}}{3}e^{+iE_3t/16}|421\rangle\right)$

$= \frac{1}{9}e^{+iE_Rt/9\hbar} - i\frac{8}{9}e^{+iE_Rt/16\hbar} = 0$

$e^{+iE_Rt/9\hbar} = 8i\, e^{+iE_Rt/16\hbar}$

$e^{+i\frac{7}{144}E_Rt/\hbar} = 8i$ impossible for real t

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(my mistake here)

Problem 4 (20 points)

The Hamiltonian for a spin-1 system in a certain basis is given by the matrix $\hat{H} = \begin{pmatrix} 7 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 7 \end{pmatrix}$, where the units are eV.

(a) What are the possible outcomes if the energy of this system is measured?
(b) What are the stationary states of this system, as represented in this basis??

(c) Does this Hamiltonain commute with $\hat{S}_z$ matrix $\hat{S}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$?

(d) Based on your answer in (c), do you expect to find the eigenstates of $\hat{S}_z$ to be stationary states?

5pnt a) $H|\alpha\rangle = E|\alpha\rangle \quad \Rightarrow \det|\hat{H} - E\cdot\hat{I}| = 0$

$$\begin{vmatrix} 7-E & 0 & 3 \\ 0 & 2-E & 0 \\ 3 & 0 & 7-E \end{vmatrix} = 0 \qquad (2-E)\left[(7-E)^2 - 9\right] = 0$$

$E_1 = 2\,eV, \quad E_2 = 10\,eV, \quad E_3 = 4\,eV$

7pnt b) $|\alpha_1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad E_1 = 2\,eV \qquad |\alpha_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad E_3 = 4\,eV$

$|\alpha_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad E_2 = 10\,eV$

5pnts c) $\hat{H}\hat{S}_z = \begin{pmatrix} 7 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 7 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 7 & 0 & -3 \\ 0 & 0 & 0 \\ 3 & 0 & 7 \end{pmatrix}$

$\hat{S}_z\hat{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 7 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 7 \end{pmatrix}$

$[\hat{H}, \hat{S}_z] = \hat{H}\hat{S}_z - \hat{S}_z\hat{H} = \begin{pmatrix} 0 & 0 & -6 \\ 0 & 0 & 0 \\ 6 & 0 & 0 \end{pmatrix}$

3pnts d) No, since non-commuting operators cannot have common basis.

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Problem 5 (15 points)
A particle of mass m is in the first excited state $|1\rangle$ of a simple harmonic oscillator with frequency ω. Calculate average value $\langle \hat{x} \rangle$ and the uncertainty Δx of the particle position.
The uncertainty of an operator is defined as $\Delta A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$.

Option 1 — operator-based

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^+)$$

$$\langle x \rangle = \langle 1|\hat{x}|1\rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\langle 1|\hat{a}|1\rangle + \langle 1|\hat{a}^+|1\rangle \right) = 0$$

$$\langle x^2 \rangle = \left(\frac{\hbar}{2m\omega}\right) \langle 1|(\hat{a}+\hat{a}^+)(\hat{a}+\hat{a}^+)|1\rangle = \left(\frac{\hbar}{2m\omega}\right)\Big(\underbrace{\langle 1|\hat{a}\hat{a}^+|1\rangle}_{(\sqrt{2})^2} + \underbrace{\langle 1|\hat{a}^+\hat{a}|1\rangle}_{1} \Big)$$

$$\hat{a}^2 + (\hat{a}^+)^2 + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a}$$

$$= \frac{3\hbar}{2m\omega}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{3\hbar}{2m\omega}}$$

Option 2 — wave-function — based

$$\langle x \rangle = \int_{-\infty}^{+\infty} x\, |\psi_1(x)|^2 dx = \sqrt{\frac{m\omega}{\pi\hbar}}\, \frac{2m\omega}{\hbar} \int_{-\infty}^{+\infty} x^3 e^{-\frac{m\omega}{\hbar}x^2} dx = 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 |\psi_1(x)|^2 dx = \sqrt{\frac{m\omega}{\pi\hbar}}\, \frac{2m\omega}{\hbar} \int_{-\infty}^{+\infty} x^4 e^{-\frac{m\omega}{\hbar}x^2} dx =$$

$$= \sqrt{\frac{\cancel{m\omega}}{\cancel{\pi}\hbar}}\, \frac{\cancel{2}m\omega}{\cancel{\hbar}} \sqrt{\frac{\cancel{\pi}\hbar}{m\omega}}\, \frac{3!!}{\cancel{4}\,2} \left(\frac{\hbar}{m\omega}\right)^2 = \frac{3\hbar}{2m\omega}$$

$3!! = 3$

$n!! = n(n-2)(n-4)\ldots$

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Problem 6 (10 points)
Two spin-1/2 particles are prepared in an entangled state:
$\frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)$, where $|\uparrow\rangle_{1,2}$ is correspondingly $|\pm z\rangle$ state for each particle.
(a) If the z component of the total spin $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$ is measured, what are the possible outcomes?

The two particles are separated, and the state of the first particle is measured using a Stern-Gerlach apparatus tilted by 60° with respect to the z axis. In a particular measurement, the first particle emerges from the positive output, so that its state is $|+60°\rangle_1 = \frac{\sqrt{3}}{2}|\uparrow\rangle_1 + \frac{1}{2}|\downarrow\rangle_1$.
(b) After the first particle measurement is made, the second particle passes through a Stern-Gerlach apparatus oriented in z direction. What is the probability it emerges from the positive output?
(c) If the Stern-Garlach apparatus for the second particle is oriented along y axis, what are the probabilities for the each output in this case?

a) Since two spins are aligned, the total S_z is either $\hbar$ or $-\hbar$

2pnts

If the first particle is measured @ $|+60°\rangle$, then the second particle is in the identical state

4pnts b) $P_{+z} = |\langle +z|+60°\rangle|^2 = \frac{3}{4}$

4pnts c) $P_{+y} = |\langle +y|+60°\rangle|^2 = \left| \frac{1}{\sqrt{2}}(1\ -i)\begin{pmatrix}\sqrt{3}/2 \\ 1/2\end{pmatrix}\right|^2 =$

$= \frac{1}{2}\left| \sqrt{3}/2 - i/2 \right|^2 = 1/2$

$P_{-y} = |\langle -y|+60°\rangle|^2 = \left| \frac{1}{\sqrt{2}}(1\ i)\begin{pmatrix}\sqrt{3}/2 \\ 1/2\end{pmatrix}\right|^2 =$

$= \frac{1}{2}\left| \sqrt{3}/2 - i/2 \right|^2 = 1/2$