Problem 1 (15 points)

Consider a particle with mass m and energy $E > 0$ that experiences a one-dimensional potential given by $V(x) = \begin{cases} 0 & x \geq 0 \\ -V_0 & x < 0 \end{cases}$

If the particle is incident from the left (i.e. from $x = -\infty$), what is the value of V_0 such that there is a 25% probability that the particle will "reflect" back?

Show all work to receive credit, and circle your final answers. This exam is closed book, but you can use the index card with reference information that you have prepared.

Problem 2 (20 points)

A quantum state of an electron spin in z bases at $t = 0$ can be written as: $|\alpha\rangle = A\begin{pmatrix}1-i\\ \sqrt{3}\end{pmatrix}$;

(a) Determine A by normalizing this state.
(b) If you measured S_z of this electron, what possible values could you get, and with what probabilities?
(c) If spin in y direction is measured, what is the probability to get the value $-\hbar/2$?
(d) What is the expectation value of $\hat{S}_x + \hat{S}_y$?
(e) As a part of the quantum circuit, the electron is sent through the phase gates $\hat{\Phi} = \begin{pmatrix}e^{i\pi/4} & 0\\ 0 & 1\end{pmatrix}$, followed by the modified Hadamar gate $\hat{H}_m = \begin{pmatrix}\sqrt{2/5} & \sqrt{3/5}\\ \sqrt{3/5} & -\sqrt{2/5}\end{pmatrix}$. Calculate the output state.

Problem 3 (20 points)

A hydrogen atom at time $t = 0$ is found in the state $|\alpha\rangle = \frac{1}{3}|300\rangle - i\frac{2\sqrt{2}}{3}|421\rangle$ (here we use the nomenclature $|nlm\rangle$ for a quantum state with quantum numbers n, l, m.

(a) Is this a stationary state? Explain your answer.

(b) What possible values of E, L^2 and L_z could arise from a measurement of this atom?

(c) What are the expectation values for measurements of each of E, L^2 and L_z?

(d) Write the time-evolved expression for this state $|\alpha(t)\rangle$.

(e) At what time this state becomes orthogonal to $|\beta\rangle = \frac{1}{3}|300\rangle + \frac{2\sqrt{2}}{3}|421\rangle$. You can assume that $\{|nlm\rangle\}$ basis is orthonormal.

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Problem 4 (20 points)

The Hamiltonian for a spin-1 system, represented in some basis, is given by the matrix $\hat{H} = \begin{pmatrix} 7 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 7 \end{pmatrix}$, where the units are eV.

(a) What are the possible values of an energy measurement of this system?
(b) What are the stationary states of this system, as represented in this basis??
(c) Does this Hamiltonain commute with $\hat{S}_z$ matrix $\hat{S}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$?
(d) Based on your answer in (c), do you expect to find the eigenstates of $\hat{S}_z$ to be stationary states?

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Problem 5 (15 points)

A particle of mass m is in the first excited state $|1\rangle$ of a simple harmonic oscillator with frequency ω. Calculate average value $\langle \hat{x} \rangle$ and the uncertainty Δx of the particle position.

The uncertainty of an operator is defined as $\Delta A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$.

Problem 6 (10 points)
Two spin-1/2 particles are prepared in an entangled state:
$\frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)$.

(a) If the z component of the total spin $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$ is measured, what are the possible outcomes?

The two particles are separated, and the state of the first particle is measured using a Stern-Gerlach apparatus tilted by 60^o with respect to the z axis. In a particular measurement, the particle emerges from the positive output, so that its state is $|+60^o\rangle_1 = \frac{\sqrt{3}}{2}|\uparrow\rangle_1 + \frac{1}{2}|\downarrow\rangle_1$.

(b) After the first particle measurement is made, the second particle passes through a Stern-Gerlach apparatus oriented in z direction. What is the probability it emerges from the positive output?

(c) If instead the Stern-Garlach apparatus for the second particle is oriented along y axis, what are the probabilities for the each output in this case?

And that's it! Congratulations, you are done with Quantum 1!

What's your exit quantum state? Add appropriate coefficients.

$$|😶\rangle = \quad |😃\rangle + \quad |🙁\rangle$$

Potentially useful information

Spin-1/2 particle

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\ \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\ \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

Eigenstates for the spin operators:

$$|+z\rangle = \begin{pmatrix}1\\0\end{pmatrix};\ |-z\rangle = \begin{pmatrix}0\\1\end{pmatrix};\ |+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix};\ |-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix};\ |+y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix};\ |-y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

Spin-1 particle

$$\hat{S}_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\ \hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\ \hat{S}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}.$$

Eigenstates of the $\hat{S}_z$ operator (in the z-basis):

$$|1,1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix};\ |1,0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix};\ |1,-1\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

The commutator of two operators is defined as $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$.

Dirac delta function

$$\int_a^b \delta(x-x_0)dx = \begin{cases}1 & a \le x_0 \le b\\ 0 & \text{otherwise}\end{cases}$$

$$\int_a^b \delta(x-x_0)f(x)dx = \begin{cases}f(x_0) & a \le x_0 \le b\\ 0 & \text{otherwise}\end{cases}$$

Kronecker delta symbol: $\delta_{nk} = \begin{cases}1 & n = k\\ 0 & n \ne k\end{cases}$

Differential equations:

$\dfrac{d^2y}{dx^2} = -k^2y$, possible solutions $y_{1,2} = \sin(kx)$ and $\cos(kx)$ or $y_{1,2} = e^{\pm ikx}$

$\dfrac{d^2y}{dx^2} = \kappa^2y$, possible solutions $y_{1,2} = e^{\pm\kappa x}$

Orthogonality of the trigonometric functions:

$\int_0^L \sin\frac{\pi nx}{L}\sin\frac{\pi kx}{L}dx = \frac{L}{2}\delta_{nk}$,

$\int_0^L \cos\frac{\pi nx}{L}\cos\frac{\pi kx}{L}dx = \frac{L}{2}\delta_{nk}$,

$\int_0^L \sin\frac{\pi nx}{L}\cos\frac{\pi kx}{L}dx = 0$

Potentially useful mathematical expressions

$$i\cdot i = -1;\ i\cdot(-i) = 1;\ 1/i = -i;$$
$$e^{i\phi} = \cos\phi + i\sin\phi;\ \cos\phi = (e^{i\phi}+e^{-i\phi})/2;\ \sin\phi = (e^{i\phi}-e^{-i\phi})/2i; \left|e^{i\phi}\right|^2 = 1;$$
$$\cos 2\phi = \cos^2\phi - \sin^2\phi;\ \sin 2\phi = 2\sin\phi\cos\phi$$

Simple Harmonic Oscillator (SHO) wave functions:

$\psi_0(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}}e^{-\frac{m\omega}{2\hbar}x^2},\ E_0 = \frac{1}{2}\hbar\omega$

$\psi_1(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{2m\omega}{\hbar}}xe^{-\frac{m\omega}{2\hbar}x^2},\ E_1 = \frac{3}{2}\hbar\omega$

$\psi_2(x) = \sqrt[4]{\frac{m\omega}{4\pi\hbar}}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega}{2\hbar}x^2},\ E_2 = \frac{5}{2}\hbar\omega$

SHO raising and lowering operators: $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x}\sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{1}{\sqrt{2m\omega\hbar}}\hat{p};\ \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right);\quad \hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a}-\hat{a}^\dagger\right)$

$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle,\ \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$