Physics 313 Final Exam
December 15, 2022

Name (please print): solutions

This test is administered under the rules and regulations of the honor system of the College of William & Mary.

Signature: ______________________________

1	2	3	4	5	6

Final score: ___________

Show all work to receive credit, and circle your final answers. This exam is closed book, but you can use a prepared index card with reference information that you have prepared.

Problem 1 (15 points)
The magnetic field is in the $x-z$ plane $\vec{B} = 3B_0\vec{i} + 4B_0\vec{k}$. The magnetic moment for an electron is $\vec{\mu} = -\gamma\vec{S}$ (here γ is a gyromagnetic ratio), and its potential energy in the magnetic field is $-\vec{\mu}\cdot\vec{B}$. For convenience you may use $\omega_0 = \gamma B_0$. (a) Write the matrix representation of this Hamiltonian in the z-basis.
(b) Find eigenvalues and eigenvectors of this Hamiltonain.
(c) Rewrite the Hamiltonian in the basis of its eigenvectors.

electron is spin 1/2 particle

6pnts a) $\hat{H} = -\hat{\vec{\mu}}\cdot\vec{B} = \gamma\hat{\vec{S}}\cdot\vec{B} = \gamma(\hat{S}_x B_x + \hat{S}_z B_z) = \gamma B_0 \frac{\hbar}{2}\left(3\begin{pmatrix}0&1\\1&0\end{pmatrix} + 4\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)$

$$\hat{H} = \frac{\cancel{3}\hbar\gamma B_0}{2}\begin{pmatrix}4&3\\3&-4\end{pmatrix}$$

6pnts b) $\hat{H}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \frac{\hbar\gamma B_0}{2}\lambda\begin{pmatrix}c_1\\c_2\end{pmatrix}$ $\quad \begin{vmatrix}4-\lambda & 3\\ 3 & -4-\lambda\end{vmatrix} = 0 \quad \lambda = \pm 5$

Eigenvalues $E_\pm = \pm\frac{5\hbar\gamma B_0}{2}$ $\quad |+\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix}3\\1\end{pmatrix}$

$|-\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix}1\\-3\end{pmatrix}$

2pnts c) $\hat{H}\Big|_{|\pm\rangle \text{ basis}} = \frac{5\hbar\gamma B_0}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

~~The~~ Any matrix is diagonal in the basis of its eigenvectors

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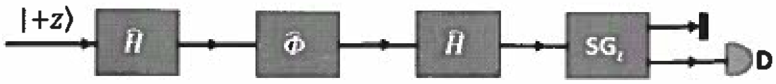
Problem 2 (20 points)
Interferometers are powerful tools for measurements and manipulations of quantum states. A possible interferometer design for a spin-1/2 particle is shown in the figure below and consists of two Hadamard gates, separated by a phase gate. In the $|\pm z\rangle$ basis the Hadamard gate operator is $\hat{\mathcal{H}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ and the phase gate operator is $\hat{\Phi} = \begin{pmatrix}e^{i\phi/2} & 0\\ 0 & e^{-i\phi/2}\end{pmatrix}$. After the interferometer the negative output of the z-oriented Stern-Gerlach apparatus is monitored using a detector D.



(a) If a particle enters the interfrometer in $|+z\rangle$ state, what is its probability to be detected at D as a function of the phase ϕ? Can we use such measurements to determine the value of ϕ?
(b) Accidentally, the initial particle spin orientation was set to be in $+x$ direction. What is the probability for measure $S_z = -\hbar/2$ at D in this case? Can this be a useful arrangement for a phase ϕ measurement?
(c) If a particle is initially in the $|+z\rangle$ state, what is its state after the first Hadamard gate?
(d) You are charged to build a physical version of the phase gate $\hat{\Phi}$ by creating a region of a constant magnetic field. Should you orient this field in x or in z direction? Explain your choice.

a) Before detection $|out\rangle = \hat{H}\hat{\Phi}\hat{H}|+z\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}e^{i\varphi/2} & 0\\ 0 & e^{-i\varphi/2}\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}|+z\rangle =$

10pnts $= \begin{pmatrix}\cos\varphi/2 & i\sin\varphi/2\\ i\sin\varphi/2 & \cos\varphi/2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}\cos\varphi/2\\ i\sin\varphi/2\end{pmatrix}$

$P_{-z} = |i\sin\varphi/2|^2 = \sin^2\varphi/2$

Yes, the probability depends on φ and can (and does) serve to measure φ

6pnts b) If $|+x\rangle$ is inserted $|out\rangle = \begin{pmatrix}\cos\varphi/2 & i\sin\varphi/2\\ i\sin\varphi/2 & \cos\varphi/2\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix} = \frac{\cos\varphi/2 + i\sin\varphi/2}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = e^{i\varphi/2}\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$

$P_{-z} = |\langle -z|out\rangle|^2 = \left|\frac{e^{i\varphi/2}}{\sqrt{2}}\right|^2 = \frac{1}{2}$

The output does not depend on φ, not a good detector

2pnts c) $|out\,1\rangle = \hat{H}|+z\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$

2pnts d) To induce opposite phases for $|+z\rangle$ and $|-z\rangle$, $\vec{B}$ must be along z

$\hat{\Phi}|+z\rangle = e^{i\varphi/2}|+z\rangle$
$\hat{\Phi}|-z\rangle = e^{-i\varphi/2}|-z\rangle$

Problem 3 (20 points)

A particle of mass m in a potential well $V(x) = \begin{cases} 0 & 0 \le x \le L \\ \infty & \text{elsewhere} \end{cases}$ is in the initial state $\psi(x) = Ax(L-x)$.

(a) Find the value of A (assume it is a real number).

(b) What is the average energy of the particle?

(c) Verify that the uncertainty principle $\Delta x \Delta p \ge \hbar/2$ is valid for this state. *Reminder:* the uncertainty for an operator $\hat{A}$ measured in a given state is defined as $\Delta A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$.

(d) A clueless experimentalist spent many days measuring probabilities of finding the particle in various energy states n of this potential well, just to "discover" that for many states this probability is zero. Show how smart you are by instantly (or after writing an integral or two) predicting what are these states. Make sure to explain it clearly here, so that even your professor understands.

5pnts a) $$\int_0^L |\psi(x)|^2 dx = 1 = A^2 \int_0^L x^2(L-x)^2 dx = A^2 L^5 \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) = \frac{A^2 \cdot L^5}{30}$$

$$A = \sqrt{30/L^5}$$

5pnts b) $$\hat{H} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

$$\langle E \rangle = \int_0^L \psi^*(x)\hat{H}\psi(x)\,dx = A^2 \int_0^L x(L-x)\left(\frac{\hbar^2}{m}\right)dx = A^2 \frac{\hbar^2}{m} \cdot L^3 \left(\frac{1}{2} - \frac{1}{3}\right) =$$

$$= \frac{30}{L^5} \cdot \frac{\hbar^2}{m} \cdot L^3 \cdot \frac{1}{6} = \frac{5\hbar^2}{mL^2}$$

8pnts c) $$\langle x \rangle = L/2 \qquad \langle x^2 \rangle = \int_0^L x^2 |\psi(x)|^2 dx = A^2 \int_0^L x^4 (L-x)^2 dx = A^2 L^7 \left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right) =$$

$$= \frac{30}{L^5} \cdot L^7 \cdot \frac{1}{105} = \frac{2}{7}L^2$$

$$\Delta x = \sqrt{\frac{2L^2}{7} - \frac{L^2}{4}} = \frac{L}{\sqrt{28}}$$

$$\langle p \rangle = 0 \qquad \langle p^2 \rangle = 2m\langle E \rangle = \frac{5\hbar^2}{L^2} \qquad \Delta p = \sqrt{10} \cdot \frac{\hbar}{L}$$

$$\Delta x \cdot \Delta p = \sqrt{\frac{5}{14}}\,\hbar > \frac{1}{2}\hbar$$

2pnts d) $$P_n = \int_0^L \psi_n(x) \cdot \psi(x)\,dx \propto \int_0^L \sin\frac{\pi n x}{L}\, \underbrace{x(L-x)}_{\text{symmetric}}\,dx$$

symmetric (n - odd) or antisymmetric (n - even)

$$\boxed{P_n = 0 \text{ for even } n}$$

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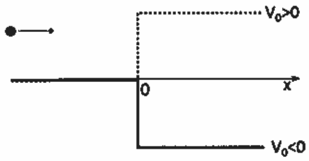
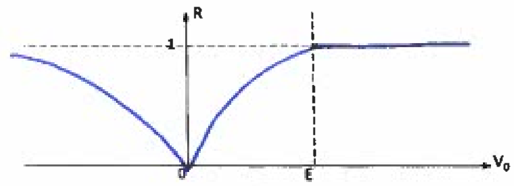
Problem 4 (15 points)

A particle of mass m moves with the energy $E > 0$ from negative infinity in $+x$ direction and reflects off a potential step $V(x) = \begin{cases} 0 & x \le 0 \\ V_0 & x > 0 \end{cases}$ as shown below.

The height of the potential step V_0 is slowly adjusted from large negative to large positive value, exceeding the particle energy E. Calculate the reflection coefficient R of such step as a function of V_0 and sketch it on the graph below. Please make sure to comment on exact values of the reflection coefficients in two marked points $V_0 = 0$ and $V_0 = E$.





5pnts

if $V_0 > E$ $R = 1$

if $V_0 < E$:

$x<0$ $\psi_< = Ae^{ikx} + Be^{-ikx}$ $k = \sqrt{\frac{2mE}{\hbar^2}}$

$x>0$ $\psi_> = Ce^{ik_1 x}$ $k_1 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

Boundary conditions: $\psi_<(0) = \psi_>(0)$ $A+B = C$

$\psi'_<(0) = \psi'_>(0)$ $ik(A-B) = ik_1 C = ik_1(A+B)$

$$B = \frac{k-k_1}{k+k_1} A$$

$$\left|\frac{B}{A}\right|^2 = R = \left(\frac{k-k_1}{k+k_1}\right)^2$$ 8pnts

$V_0 = 0$ $k_1 = k$ $R = 0$ (no step)

$V_0 = E$ $k_1 = 0$ $R = 1$ 2pnts

$V_0 \to -\infty$ $k_1 \to \infty$ $R \to 1$

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Problem 5 (20 points)

Let us consider a rigid rotator with a Hamiltonian $\hat{H} = \hat{L}^2/2I$ (where I is its moment of inertia). We define kets $|l,m\rangle$ as eigenstates of the angular momentum operators such as $\hat{L}^2|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle$ and $\hat{L}_z|l,m\rangle = \hbar m|l,m\rangle$.

(a) In a particular measurement the energy of the particle was found to be $6\hbar^2/I$. List all possible quantum states for this particle.

(b) In a different measurement the particle is prepared in the following superposition: $|\psi\rangle = \frac{1}{\sqrt{5}}|3,2\rangle - \frac{2}{\sqrt{5}}|1,0\rangle$. Find $|\psi(t)\rangle$.

(c) What are the average values of $\hat{L}^2$ and $\hat{L}_z$ in this state as functions of time?

(d) Suppose that this rigid rotator is immersed in a uniform magnetic field, such that it experiences the quadratic Zeeman effect. In this case the Hamiltonian becomes $\hat{H}_1 = \hat{L}^2/2I + \alpha(\hat{L}^2 - \hat{L}_z^2)$.

Show that $|l,m\rangle$ states are eigenstates of this Hamiltonian and find their energy eigenvalues. If a particle is in a state with a known l, how many possible energy values one can measure? Does this magnetic field lift the m-state degeneracies fully or only partially?

5pnts a) $\hat{H}|l,m\rangle = \frac{\hbar^2 l(l+1)}{2I}$ $\quad = \frac{\hbar^2 l(l+1)}{2I} = 6\frac{\hbar^2}{I}$ $\quad l(l+1) = 12$, $l = 3$

Possible states $|3,0\rangle, |3,\pm1\rangle, |3,\pm2\rangle$, $|3,\pm3\rangle$

$l = 3 \quad m = 0, \pm1, \pm2, \pm3$

5pnts b) $|\psi\rangle = \frac{1}{\sqrt{5}}|3,2\rangle - \frac{2}{\sqrt{5}}|1,0\rangle$

$E_{l=3} = \frac{6\hbar^2}{I} \quad E\cdot t/\hbar = \frac{6\hbar t}{I} \quad E_{l=1} = \frac{\hbar^2}{I} \quad Et/\hbar = \hbar t/I$

$|\psi(t)\rangle = \frac{1}{\sqrt{5}} e^{-6\hbar t/I}|3,2\rangle - \frac{2}{\sqrt{5}} e^{-\hbar t/I}|1,0\rangle$

5pnts c) $\langle L^2\rangle = \langle\psi(t)|\hat{L}^2|\psi(t)\rangle = \frac{1}{5}\hbar^2\cdot 12 + \frac{4}{5}\hbar^2\cdot 2 = 4\hbar^2$

$\langle L_z\rangle = \langle\psi(t)|\hat{L}_z|\psi(t)\rangle = \frac{1}{5}\cdot 2\hbar = \frac{2}{5}\hbar$

Both don't depend on time, since $|l,m\rangle$ are their eigenstates

5pnts d) $\hat{H}_1 = \frac{\hat{L}^2}{2I} + \alpha(\hat{L}^2 - \hat{L}_z^2)$

$\hat{H}_1|l,m\rangle = \frac{\hbar^2}{2I} l(l+1)|l,m\rangle + \alpha\hbar^2(l(l+1) - m^2)|l,m\rangle = \left[\hbar^2 l(l+1)\left(\frac{1}{2I} - \alpha\right) - \alpha\hbar^2 m^2\right] \times |l,m\rangle$

Eigen states, $E_{lm} = \hbar^2 l(l+1)\left(\frac{1}{2I} - \alpha\right) - \hbar^2\alpha m^2$

State with different $|m|$ have different energy,

for $l = 2$ $\quad m = 0, \pm1, \pm2 \rightarrow$ 3 possible energies

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$|l,\pm m\rangle$ states have the same energy, for $m \neq 0$

each energy state is still twice degenerate

Problem 6 (10 points)

Reminder: Rydberg energy $E_R = \frac{m_e(ke^2)^2}{2\hbar^2} = 13.6$ eV.

(a) After repeating the energy measurements for identical hydrogen atoms, a physicist established that in 9% cases it has energy -13.6 eV, in 64% cases its energy is -1.51 eV, and in remaining 27% it is -0.136 eV. If its orbital angular momentum is measured, it is always zero. Write down a possible quantum state for such an atom. Does the provided information fully characterize the state, or do you have some freedom to choose how you define it?

(b) Cross out the combinations of $\{n, l, m\}$ numbers that result in unphysical quantum states $|n, l, m\rangle$:

{3,2,2}	~~{2,2,0}~~	~~{3,0,1}~~	{3,1,-1}	~~{3,-2,2}~~

5pnts a) $E_1 = -13.6\,eV = -E_R, n = 1$ $\quad P_1 = 0.09 \quad |C_1| = \sqrt{P_1} = 0.3$

$E_2 = -1.51\,eV = -\frac{E_R}{9}, n = 3$ $\quad P_2 = 0.64 \quad |C_2| = \sqrt{P_2} = 0.8$

$E_3 = -0.136\,eV = -\frac{E_3}{100}, n = 10$ $\quad P_3 = 0.27 \quad |C_3| = \sqrt{P_3} \approx 0.52$

We cannot know anything about phases of the coefficien

and can choose as we please

$l = 0, m = 0$

$|\psi\rangle = 0.3\,|1,0,0\rangle - 0.8i\,|3,0,0\rangle + 0.52\,e^{i\pi/8}\,|10,0,0\rangle$

5pnts b) $l = 0, 1 \ldots n-1$

$m = 0, \pm 1, \ldots \pm l$

~~[illegible]~~ unphysical

$\{2,2,0\} \longrightarrow n = l$

$\{3,0,1\} \longrightarrow m > l$