

Physics 313 Final Exam

December 15, 2022

Name (please print): _____

This test is administered under the rules and regulations of the honor system of the College of William & Mary.

Signature: _____

1	2	3	4	5	6

Final score: _____

Show all work to receive credit, and circle your final answers. This exam is closed book, but you can use a prepared index card with reference information that you have prepared.

Problem 1 (15 points)

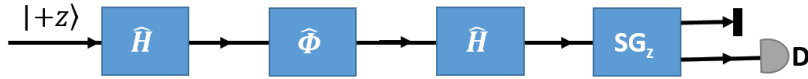
The magnetic field is in the $x - z$ plane $\vec{B} = 3B_0\vec{i} + 4B_0\vec{k}$. The magnetic moment for an electron is $\vec{\mu} = -\gamma\vec{S}$ (here γ is a gyromagnetic ratio), and its potential energy in the magnetic field is $-\vec{\mu} \cdot \vec{B}$.

- (a) Write the matrix representation of this Hamiltonian in the z -basis.
- (b) Find eigenvalues and eigenvectors of this Hamiltonian.
- (c) Rewrite the Hamiltonian in the basis of its eigenvectors.

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Problem 2 (20 points)

Interferometers are powerful tools for measurements and manipulations of quantum states. A possible interferometer design for a spin-1/2 particle is shown in the figure below and consists of two Hadamard gates, separated by a phase gate. In the $|\pm z\rangle$ basis the Hadamard gate operator is $\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and the phase gate operator is $\hat{\Phi} = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}$. After the interferometer the negative output of the z -oriented Stern-Gerlach apparatus is monitored using a detector D .



- If a particle enters the interferometer in $|+z\rangle$ state, what is its probability to be detected at D as a function of the phase ϕ ? Can we use such measurements to determine the value of ϕ ?
- Accidentally, the initial particle spin orientation was set to be in $+x$ direction. What is the probability for measure $S_z = -\hbar/2$ at D in this case? Can this be a useful arrangement for a phase ϕ measurement?
- If a particle is initially in the $|+z\rangle$ state, what is its state after the first Hadamard gate?
- You are charged to build a physical version of the phase gate $\hat{\Phi}$ by creating a region of a constant magnetic field. Should you orient this field in x or in z direction? Explain your choice.

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Problem 3 (20 points)

A particle of mass m in a potential well $V(x) = \begin{cases} 0 & 0 \leq x \leq L \\ \infty & \text{elsewhere} \end{cases}$ is in the initial state $\psi(x) = Ax(L - x)$.

- (a) Find the value of A (assume it is a real number).
- (b) What is the average energy of the particle?
- (c) Verify that the position-momentum uncertainty principle is valid for this state. *Reminder:* the uncertainty for an operator \hat{A} measured in a given state is defined as $\Delta A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$.
- (d) A clueless experimentalist spent many days measuring probabilities of finding the particle in various energy states n of this potential well, just to “discover” that for many states this probability is zero. Show how smart you are by instantly (or after writing an integral or two) predicting what are these states. *Hint: think of a symmetry of various eigenstates.*

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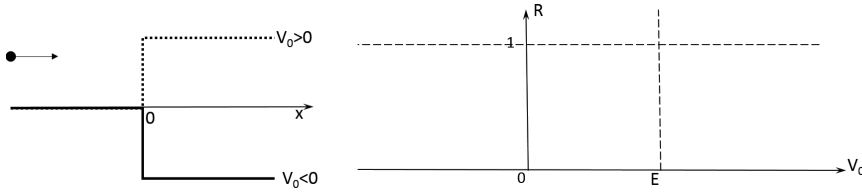
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Problem 4 (15 points)

A particle of mass m moves with the energy $E > 0$ from negative infinity in $+x$ direction and reflects off a potential step

$$V(x) = \begin{cases} 0 & x \leq 0 \\ V_0 & x > 0 \end{cases} \text{ as shown below.}$$

The height of the potential step V_0 is slowly adjusted from large negative to large positive value, exceeding the particle energy E . Calculate the reflection coefficient R of such step as a function of V_0 and sketch it on the graph below. Please make sure to comment on exact values of the reflection coefficients in two marked points $V_0 = 0$ and $V_0 = E$.



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Problem 5 (20 points)

Let us consider a rigid rotator with a Hamiltonian $\hat{H} = \hat{L}^2/2I$ (where I is its moment of inertia). We define kets $|l, m\rangle$ as eigenstates of the angular momentum operators such as $\hat{L}^2|l, m\rangle = \hbar^2 l(l+1)|l, m\rangle$ and $\hat{L}_z|l, m\rangle = \hbar m|l, m\rangle$.

(a) In a particular measurement the energy of the particle was found to be $6\hbar^2/I$. List all possible quantum states for this particle.

(b) In a different measurement the particle is prepared in the following superposition: $|\psi\rangle = \frac{1}{\sqrt{5}}|3, 2\rangle - \frac{2}{\sqrt{5}}|1, 0\rangle$. Find $|\psi(t)\rangle$.

(c) What are the average values of \hat{L}^2 and \hat{L}_z in this state as functions of time?

(d) Suppose that this rigid rotator is immersed in a uniform magnetic field, such that it experiences the quadratic Zeeman effect.

In this case the Hamiltonian becomes $\hat{H}_1 = \hat{L}^2/2I + \hbar^2\alpha(\hat{L}^2 - \hat{L}_z^2)$.

Show that $|l, m\rangle$ states are eigenstates of this Hamiltonian and find their energy eigenvalues. If a particle is in a state with $l = 2$, how many possible energy values it can have? Does such interaction lift the m -state degeneracies fully or only partially?

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Problem 6 (10 points)

Reminder: Rydberg energy $E_R = \frac{m_e(k_e^2)^2}{2\hbar^2} = 13.6 \text{ eV}$.

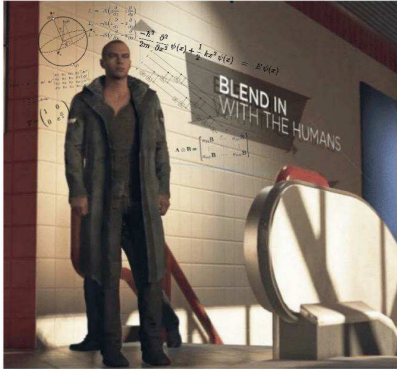
(a) After repeating the energy measurements for identical hydrogen atoms, a physicist established that in 9% cases it has energy -13.6 eV , in 64% cases its energy is -1.51 eV , and in remaining 27% it is -0.136 eV . If its orbital angular momentum is measured, it is always zero. Write down a possible quantum state for such an atom. Does the provided information fully characterize the state, or do you have some freedom to choose how you define it?

(b) Cross out the combinations of $\{n, l, m\}$ numbers that result in unphysical quantum states $|n, l, m\rangle$:

$\{3,2,2\}$	$\{2,2,0\}$	$\{3,0,1\}$	$\{3,1,-1\}$	$\{3,-2,2\}$
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Best of luck! You can do it!

When you finally finish your
quantum exam



Potentially useful information

Spin-1/2 particle

$$\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Eigenstates for the spin operators:

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad |+x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad |-x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \quad |+y\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}; \quad |-y\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Spin-1 particle

$$\hat{S}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \hat{S}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \hat{S}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

Eigenstates of the \hat{S}_z operator (in z -basis):

$$|1, 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad |1, 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad |1, -1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Dirac delta function

$$\int_a^b \delta(x - x_0) dx = \begin{cases} 1 & a \leq x_0 \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$\int_a^b \delta(x - x_0) f(x) dx = \begin{cases} f(x_0) & a \leq x_0 \leq b \\ 0 & \text{otherwise} \end{cases}$$

Kronecker delta symbol:

$$\delta_{nk} = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$$

Differential equations:

$$\frac{d^2 y}{dx^2} = -k^2 y, \text{ possible solutions } y_{1,2} = \sin(kx) \text{ and } \cos(kx) \text{ or } y_{1,2} = e^{\pm ikx}$$

$$\frac{d^2 y}{dx^2} = \kappa^2 y, \text{ possible solutions } y_{1,2} = e^{\pm \kappa x}$$

Orthogonality of the trigonometric functions:

$$\int_0^L \sin \frac{\pi n x}{L} \sin \frac{\pi k x}{L} dx = \frac{L}{2} \delta_{nk},$$

$$\int_0^L \cos \frac{\pi n x}{L} \cos \frac{\pi k x}{L} dx = \frac{L}{2} \delta_{nk},$$

$$\int_0^L \sin \frac{\pi n x}{L} \cos \frac{\pi k x}{L} dx = 0$$

Expressions involving complex numbers and trig functions

$$i \cdot i = -1; \quad i \cdot (-i) = 1; \quad 1/i = -i;$$

$$e^{i\phi} = \cos \phi + i \sin \phi; \quad \cos \phi = (e^{i\phi} + e^{-i\phi})/2; \quad \sin \phi = (e^{i\phi} - e^{-i\phi})/2i;$$

$$|e^{i\phi}|^2 = 1;$$

$$\cos 2\phi = \cos^2 \phi - \sin^2 \phi; \quad \sin 2\phi = 2 \sin \phi \cos \phi$$