Maxwell's equation – crown jewel of electromagnetism

Most of the equations were known before, Maxwell put things together and analysed the results

~~In vacuum, in the absence of any electric charges~~

General form of Maxwell's eqn

$$\begin{cases} \nabla \cdot \vec{D} = \rho \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{H} = \vec{j} + \dfrac{\partial \vec{D}}{\partial t} \end{cases}$$

$\rho$ – electric charge density

$\vec{j}$ – electric current density

$\vec{E}$ – electric field ] external

$\vec{B}$ – magnetic field ] parameters

$\vec{D} = \vec{E} + \underbrace{\vec{P}}$

polarization, electrical response of the material $\vec{P} = \varepsilon_0 \chi \vec{E}$, $\vec{D} = \varepsilon_0 \varepsilon \vec{E}$

$\chi$ – susceptibility, $\varepsilon$ – dialectric constant

$\vec{H} = \vec{B}/\mu\mu_0$ $\mu$ – permeability

For all materials we care about $\mu = 1$

$\vec{H} = \vec{B}/\mu_0$

$$\vec{\nabla} = \vec{e}_x \cdot \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z}$$ gradient vector

Looking at Maxwell's equations we see that electric field can originate from electrical charges (first eqn) or time-varying magnetic field ($3^{rd}$ eqn)

Magnetic field can originate from electrical currents or time-varying electric field. ($4^{th}$ eqn)

| | | |
|---|---|---|
| Electric field varying in time and space | $\Longrightarrow$ <br> $\Longleftarrow$ | Magnetic field varying in time and space |

electro-magnetic wave

In vacuum, with no charges around

$$
\begin{cases}
\nabla \cdot \vec{E} = 0 & \nabla \times \vec{E} = -\partial \vec{B}/\partial t \\
\nabla \cdot \vec{B} = 0 & \nabla \times \vec{B} = \mu_0 \varepsilon_0 \, \partial \vec{E}/\partial t
\end{cases}
$$

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

or

$$\frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

same for $\vec{B}$

Solution: $\vec{E} = \vec{E}_0 \cos(\vec{k}\vec{r} - \omega t + \varphi)$

$\vec{k}$ – wave propagation direction

Previously we chose $\vec{k}$ to point along $z$ direction

$$\vec{E} = \vec{E}_0 \cos(kz - \omega t + \varphi)$$
$$\vec{B} = \vec{B}_0 \cos(kz - \omega t + \varphi)$$

We can now say more about the waves!

$$\frac{\partial^2 \vec{E}}{\partial z^2} = -k^2 \vec{E} \qquad \frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E}$$

$$k^2 = \mu_0 \varepsilon_0 \omega^2 \qquad c = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Speed of light in vacuum

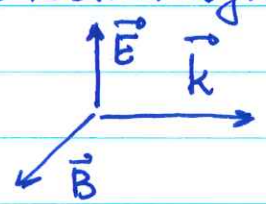
$$\nabla \cdot \vec{E} = 0 \Rightarrow \cancel{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}} + \frac{\partial E_z}{\partial z} = 0$$

$=0$ no dependence on $x, y$

$E_z$ must be $0$

$\nabla \cdot \vec{B} = 0$ $\quad B_z$ must be $0$

Electromagnetic wave is a transverse wave




$$\vec{E} \perp \vec{B} \perp \vec{k}$$

$\vec{E}$ – direction of the electric field determines the polarization of the light

For a linearly-polarized light we can define

$$E_x(z,t) = E_0 \cos(kz - \omega t + \varphi) \qquad B_0 = \frac{kE_0}{\omega} = \frac{E_0}{c}$$
$$B_y(z,t) = B_0 \cos(kz - \omega t + \varphi)$$

What about materials?

If there are free charges around, electric field can move them, producing real work. Because of the energy conservation, that means that the energy of the electromagnetic field is reduced, and light is attenuated - absorbed.
That is why light cannot propagate inside metals (too many free electrons!)

In dialectrics - no free charges
$\rho = 0 \quad \vec{j} = 0 \quad \vec{D} = \varepsilon_0 \varepsilon \vec{E}$
all responce of bound charges in the atoms / molecules is hidden inside $\varepsilon$ - dialectric constant.

Maxwell's eqns in a dialectric ($\mu = 1$)

$$\begin{cases} \nabla \cdot \vec{E} = 0 & \nabla \times \vec{E} = -\partial \vec{B}/\partial t \\ \nabla \cdot \vec{B} = 0 & \nabla \times \vec{B} = \mu_0 \varepsilon_0 \underline{\underline{\varepsilon}} \, \partial E/\partial t \end{cases}$$

same as in vacuum, but

$$k^2 = \mu_0 \varepsilon_0 \varepsilon \qquad v = \frac{1}{\sqrt{\mu_0 \varepsilon_0 \varepsilon}} = \frac{c}{\sqrt{\varepsilon}} = \frac{c}{n}$$

Refractive index $n = \sqrt{\varepsilon}$

Inside the dialectric the e-m wave propagates just as in vacuum, but slower, since now the electric field consists of not only the original wave, but also of re-radiated wave from excited bound electrons