Simple harmonic oscillator (SHO)

\[ U(x) = \frac{1}{2} kx^2 \]

Classical frequency \( \omega_0 = \sqrt{\frac{k}{m}} \)

Schrödinger equation

\[ -\frac{\hbar^2}{2m} \psi''(x) + \frac{1}{2} kx^2 \psi(x) = E \psi(x) \]

What can we guess about SHO wave functions without solving the Schrödinger equation?

1. Oscillating solutions
2. Decaying solutions
3. Number of extrema = number of the energy level
4. Since the potential is symmetric \( U(x) = U(-x) \), \( \psi(x) = \pm \psi(x) \) wavefunctions are either symmetric or antisymmetric (since \( \psi''(x) = \pm \psi''(-x) \))

Since the probability of finding a classical oscillator is higher near turning points (since it moves slower there), then \( |\psi(x)|^2 \) should be higher near the turning points than in the center for higher energy levels (due to the correspondence principle).
Ground state: \( \psi_0(x) = A e^{-\alpha x^2} \)

we guess this form

\[ \psi_0'(x) = -2d \cdot A e^{-\alpha x^2} = -2d \cdot \psi_0(x) \]

\[ \psi_0''(x) = -2d \cdot A e^{-\alpha x^2} + 4d^2 x^2 \cdot A e^{-\alpha x^2} = (-2d + 4d^2 x^2) \psi_0(x) \]

\[ -\frac{\hbar^2}{2m} (-2d + 4d^2 x^2) \psi_0(x) + \frac{1}{2} kx^2 \psi_0(x) = E_0 \psi_0(x) \]

\[ \left( \frac{\hbar^2}{2m} - E_0 \right) + \left( -2d \frac{d^2}{m} + \frac{1}{2} k \right) x^2 = 0 \]

Then

\[ \alpha = \frac{km}{\hbar^2} = \frac{\omega_0^2 m^2}{4 \hbar^2} \]

\[ \alpha = \frac{km}{2 \hbar^2} \Rightarrow E_0 = \frac{\alpha \hbar \omega_0}{2} \]

Ground state of any quantum oscillator has energy. Since in quantum electrodynamics electromagnetic waves are treated as oscillators, in vacuum (i.e., the absence of photons) corresponds to the zeroth SHO state, and contains \( \frac{1}{2} \hbar \omega_0 \) of energy.

Ground state \( E_0 = \frac{1}{2} \hbar \omega_0 \)

\[ \psi_0 = \left( \frac{\alpha \hbar}{\pi \hbar} \right)^{1/4} e^{-\alpha x^2} \]

Higher excited states, \( E_n = (n + \frac{1}{2}) \hbar \omega_0 \)

\[ \psi_n = \left( \frac{\alpha \hbar n!}{\pi \hbar^{2n+1}} \right)^{1/2} e^{-\alpha x^2} H_n \left( \sqrt{\frac{\alpha \hbar}{\hbar}} x \right) \]

\( H_n(x) \) – special Hermitian polynomials.

Distinct feature of SHO – equidistant energy spectrum.
Energy difference b/w any two consecutive states:

\[ E_{n+1} - E_n = \hbar \omega \]

Looks familiar?!

Back to black body radiation problem. Since the atoms inside crystal lattice vibrate, they can be more or less accurately described by the picture.

Thus, if all oscillators are originally in their ground state, and the energy required to excite the oscillator to the next level, one needs to provide sufficient energy \( \Delta E = \hbar \omega \).

Thus, if the energy of a black body is limited, there always will be a "cut-off" for high energy radiation (small wavelength).
Simple harmonic potential is one of the most useful in physics. Any (well, almost any) potential near equilibrium can be treated as SHO:

\[ U(x) = U(0) + \frac{dU}{dx}(0) \cdot x + \frac{d^2U}{dx^2}(0) \cdot x^2 + [\text{higher order terms}] \]

\[ \frac{dU}{dx}(0) = 0 \quad \text{SHO near minimum} \quad U(W) \propto -\frac{dU}{dx}(0) x^2 \]

Time dependent wave function:

\[ \psi_n(x,\tau) = \psi_n(x,\tau) e^{-i\frac{\pi}{\hbar}} \]

For SHO \( \varepsilon_n = (n-\frac{1}{2}) \hbar \omega \)

\[ \psi_n(x,\tau) = A \cdot \psi_n(\omega) e^{-i\frac{x^2}{\hbar}} = A \psi_n(\omega) e^{-i\omega t} \]

\[ \psi_n(x,\tau) \propto H_n \left( \sqrt{\frac{m \omega x}{\hbar}} \right) e^{\frac{-m \omega x}{\hbar}} e^{-i\omega t} \]
Time evolution of wave function

Suppose that \( \Psi_1(x) \) is a solution of Schrödinger eqn with \( E_1 \) energy level, and \( \Psi_2(x) \) for \( E_2 \) energy level.

Then

\[
\Psi_1(x,t) = \Psi_1(x) e^{-iE_1 t / \hbar}
\]

\[
\Psi_2(x,t) = \Psi_2(x) e^{-iE_2 t / \hbar}
\]

Wave functions are time dependent, but physical probabilities

\[
P_1(x,t) = |\Psi_1(x)|^2, \quad P_2(x,t) = |\Psi_2(x)|^2
\]

are time independent.

Suppose that at \( t=0 \) a particle is prepared in a quantum superposition of these two states

\[
\Psi(x) = A \Psi_1(x) + B \Psi_2(x) \quad (|A|^2 + |B|^2 = 1)
\]

So if measured, this particle is found in state 1 with probability \( |A|^2 \), and in state 2 with probability \( |B|^2 \).

How does it change in time?

\[
\Psi(x,t) = A \Psi_1(x) e^{-iE_1 t / \hbar} + B \Psi_2(x) e^{-iE_2 t / \hbar}
\]

\[
P(x,t) = |\Psi(x,t)|^2 = \Psi(x,t) \cdot \Psi^*(x,t) = (A \Psi_1(x) e^{-iE_1 t / \hbar} + B \Psi_2(x) e^{-iE_2 t / \hbar}) (A^* \Psi_1^*(x) e^{iE_1 t / \hbar} + B^* \Psi_2^*(x) e^{iE_2 t / \hbar})
\]

\[
= |A|^2 |\Psi_1(x)|^2 + |B|^2 |\Psi_2(x)|^2 + A^* B \Psi_1^* \Psi_2 \ e^{i(E_1 - E_2) t / \hbar} + A B^* \Psi_2^* \Psi_1 \ e^{-i(E_1 - E_2) t / \hbar}
\]

\[
= |A|^2 |\Psi_1(x)|^2 + |B|^2 |\Psi_2(x)|^2 + 2 \text{Re} \left[ A^* B \Psi_1^* \Psi_2 \ e^{i(E_1 - E_2) t / \hbar} \right]
\]