Central forces — potential energy depends only on the distance from the center.

Coulomb potential \( U(r) = -\frac{ke^2}{r} \)

Spherical symmetry

Schrödinger eqn in the spherical coordinates

\[
\begin{align*}
&x = r \cos \theta \sin \phi \\
y = r \sin \theta \sin \phi \\
z = r \cos \phi
\end{align*}
\]

\[
-H\frac{\hbar^2}{2m} \nabla^2 \psi \left( \vec{r} \right) + U(r) \psi \left( \vec{r} \right) = E \psi \left( \vec{r} \right)
\]

\[
-H\frac{\hbar^2}{2m} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi \left( \vec{r} \right)}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi \left( \vec{r} \right)}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi \left( \vec{r} \right)}{\partial \phi^2} \right]
\]

\[+ U(r) \psi \left( \vec{r} \right) = E \psi \left( \vec{r} \right)\]

\[
\sum \int \hat{L}^2 \psi \left( \vec{r} \right) = \hat{L}^2 \psi \left( \vec{r} \right)
\]

\( \hat{L} \) — angular momentum
Classical angular momentum

\[ \vec{\mathcal{L}} = \vec{r} \times \vec{p} \]

\[ L_z = x \cdot p_y - y \cdot p_x \]

\[ L_x = y \cdot p_z - z \cdot p_y \]

We cannot measure \( L_x \) and \( L_y \) at the same time.

Traditionally people measure

\[ \hat{\mathcal{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \]

- value of the angular momentum

\[ \hat{L}_z \] - azimuthal component of the angular momentum.

One can measure values of \( \hat{\mathcal{L}}^2 \) and \( \hat{L}_z^2 \) at the same time, but then we lose information about \( L_x \) and \( L_y \).

\[ \hat{\mathcal{L}}^2 \psi = -\frac{\hbar^2}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) - \frac{\hbar^2}{\sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \]

\[ \hat{L}_z \psi = -i \hbar \frac{\partial \psi}{\partial \phi} \]

Example: \( \hat{H} \psi = E \psi \) eigen value
If the angular part of a wave function is described by a spherical function $Y_{lm}(\theta, \phi)$, then this state has definite value for the total angular momentum

$$\langle \hat{L}^2 \rangle = \hbar^2 l(l+1)$$

and it has definite value for the $z$-component of the angular momentum

$$\langle L_z \rangle = \pm m$$

$l, m$ are integers
$l$ is positive integer $0, 1, 2, \ldots$
$m$ is positive/negative integer, but $|m| \leq l$

State with $l=2$, $m = \pm 2, \pm 1, 0$

$$\langle \hat{L}^2 \rangle = \hbar^2 l(l+1)$$

"Length" of this vector $\sqrt{\langle \hat{L}^2 \rangle} = \hbar \sqrt{l(l+1)}$

$$L_z = -l\hbar, -(l-1)\hbar, \ldots, l\hbar$$

$$\theta = \frac{\hbar L_z}{L \cdot L}$$

$m = 0$

$$\cos \theta = \frac{\hbar L_z}{\sqrt{l(l+1)}} = \frac{l}{\sqrt{l(l+1)}}$$

$l=1 \implies \theta = 45^\circ$