Einstein box paradox

\[ A \text{ photon emitted on one side with momentum } P_0 \]

Momentum conservation

\[ M_{\text{box}} \cdot V_{\text{box}} = P_0 \]

While the photon flies, the box moves by

\[ V_{\text{box}} \cdot t_{\text{flight}} = V_{\text{box}} \cdot \frac{L}{c} \]

It will stop after the photon is reabsorbed on the other side, but if the center of mass has shifted by

\[ V_{\text{box}} \cdot \frac{L}{c} = \left( \frac{P_0 \cdot L}{M_{\text{box}}} \right) > 0 \]

That violates Newton's laws!

Resolution: to emit a photon we have to convert some mass!

\[ \Delta m c^2 = E_g = P_0 \cdot c \]

When the photon travels from one side to another, it transfers \( \Delta m = P_0 / c \) from one side of the box to the other.

Change in the center of mass position

\[ \Delta x_{\text{cm}} = \frac{\Delta m}{M_{\text{box}}} \cdot \frac{L}{c} = \frac{P_0 \cdot L}{M_{\text{box}} \cdot c} \]

That compensates the photon kick!
Momentum 4-vector

Space-time 4-vector: \((ict, x, y, z)\)

Lorentz transformation:
\[
\begin{align*}
ct' &= \gamma (ct - \frac{v}{c} x) \\
x' &= \gamma (x - \frac{v}{c} (ct))
\end{align*}
\]

Space-time 4-vector momentum \((\frac{E}{c}, p_x, p_y, p_z)\) transforms the same way

\[
\begin{align*}
\frac{E'}{c} &= \gamma \left( \frac{E}{c} - \frac{v}{c} \cdot P \right) \quad \text{(consider } P = (p, 0, 0)) \\
\begin{cases}
E' = \gamma (E - v \cdot P) \\
p' = \gamma (p - \frac{v}{c^2} E')
\end{cases}
\end{align*}
\]

Lorentz invariant:
\[
\Delta S^2 = c t^2 - x^2
\]
for momentum:
\[
\Delta E_0^2 - p^2 = \text{const} = m^2 c^2
\]
(since \(E^2 = m_0^2 c^4 + p^2 c^2\))
Back to the pion production

In the rest frame of the pions

Since in this rest frame it requires minimum kinetic energy, in any other reference frame it requires min kinetic energy.

Relativistic velocity addition

\[ u' = \frac{u + v}{1 + \frac{v \cdot u}{c^2}} = \frac{2v}{1 + \frac{v^2}{c^2}} = 0.65c \quad \gamma' = 1.31 \]

Total energy of the first proton

\[ E' = \frac{m_0c^2}{\sqrt{1 - \frac{u'^2}{c^2}}} = \gamma' \cdot m_0c^2 \]

\[ K' = E' - m_0c^2 = 0.31 m_0c^2 \]

\[ K' \approx 290 \text{ MeV} \]

This reaction requires more than double of the rest energy of a pion.
Forces in relativity

Second Newton law \( \vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} \)

Lorentz force (for a charge moving in electric and magnetic fields)

\[ \vec{F}_L = q(E + \vec{v} \times \vec{B}) \]

In general, \( \frac{d\vec{p}}{dt} = \vec{F}_L \) is hard to solve, except for the case of pure magnetic field

\[ \vec{F}_B = q \vec{v} \times \vec{B} = \frac{d(m\vec{v})}{dt} \]

Since \( \vec{F}_B \perp \vec{v} \), \( \vec{v} \) does not change, \( m\vec{v} \) is constant, particle in a circular motion:

\[ m\vec{v} \frac{v^2}{R} = qvB \implies \frac{P}{R} = qB \]

\[ R = \frac{P}{qB} \]

Such "simple" situation is used in many detectors (e.g., bubble chambers) to measure a momentum for various particles.
Moving charge paradox

Electrically neutral wire
stationary lattice of ions
and moving electrons
Drift velocity \( v \sim 1 \text{ mm/s} \)

A positive charge \( q \)
moves with the same speed as electrons.

What force acts on \( q \)? Magnetic!

Current \( I = \frac{\Delta \text{charge} / \text{unit}}{\text{time}} = \lambda_0 \cdot v \); continue

where \( \lambda_0 \) - linear charge density
\( B = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 \lambda_0 v}{2\pi r} \)

Magnetic force
\( \vec{F}_B = q \cdot \vec{v} \times \vec{B} \)
up (away from)
the wire

Let's move for the RF of moving electron
\( +q \)
The test charge is stationary - no magnetic force

\( \vec{v} \)
Wire is still neutral -
no electric force

[No force ?!]

Wire is now not neutral - must
consider Lorentz contraction
(even though it is tiny)

Remember: moving object seems shorter.
In the lab frame the density of electrons (moving) and ions (stationary) are the same. In the electrons, RF ions are moving thus the distance b/w each two ions seems smaller

$$\lambda'_+ = \lambda_0 \sqrt{1 - \frac{v^2}{c^2}} \approx \lambda_0 + \lambda_0 \frac{v^2}{2c^2}$$

At the same time electrons density was measured in the RF where they were moving thus it appeared higher than in their proper rest frame

$$\lambda'_- = \frac{\lambda_0}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}} \lambda_0 \approx \lambda_0 - \lambda_0 \frac{v^2}{2c^2}$$

Thus, residual the wire in this RF has a residual electric charge density (positive):

$$\delta \lambda' = \lambda'_+ - \lambda'_- = \lambda_0 \frac{v^2}{c^2}$$

producing the electric field

$$E'_e = \frac{\delta \lambda'}{2\pi \epsilon_0} = \frac{2 \pi \epsilon_0 V^2}{\lambda_0}$$

and the electric force

$$F'_e = q_e E'_e = \frac{q_e \lambda_0 v^2}{2\pi \epsilon_0} = \frac{m_0 q_e \lambda_0 v^2}{2\pi r}$$

away from the wire

$$F'_e = F_B$$

Thus, two observers will measure the same force, but will disagree on its origin.

This example also illustrate that electric and magnetic fields are really the same physical phenomena — electro magnetism.