"Practical" rules of relativity

1. When calculating anything you must use values in the same RF. If something is measured in a different RF, you must "translate" it (use time dilation, length contraction).

2. Two observers in two RFs (i.e., moving with respect to each other) will agree on timing of any event only if they are at the same point in space.

3. Two events can be simultaneous only in one RF. They will appear to happen at two different times.

4. When applying time dilation/length contraction, we describe the "translation" for temporal or spatial intervals. Lorentz transformations describe "translation" for space and time coordinates of a single event.
Time synchronization in moving frames

Two things we have discovered last time:
1. In two reference frames in relative motion, time appears to run slower in the moving frame.
2. The distance between two points in space seems contracted in the moving frame.

\[
\Delta t = \gamma \Delta t' \\
\Delta x = \frac{1}{\gamma} \sqrt{1 - \frac{v^2}{c^2}} \cdot \Delta x'
\]

where \( \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \)

Clock synchronization

We have established that we can synchronize any clocks in the same reference frame.

Both clocks will register identical pulses of light simultaneously after

\[
\Delta t = \frac{e}{2c}
\]
The situation will change, however, if this experiment is moving.

In Bob's frame, two clocks start at the same time.

In Alice's frame, a few things are different:

1. The length of Bob's apparatus is shorter: 
   \[ L_A = \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}} \]

2. The back clock moves toward the light pulse, so it takes less time to start them.

\[ t = 0 \]
\[ t = t_B \]
\[ v_{tB} + c_{tB} = \frac{L_A}{2} = \frac{L}{2} \sqrt{1 - \frac{v^2}{c^2}} \]

\[ t_B = \frac{1}{c + v} \frac{L}{2} \sqrt{1 - \frac{v^2}{c^2}} \]

This is the time Alice measures to pass between the flash of light and the back clock starting.
Similarly, the front clock move away from the light pulse, so it takes longer for it to start:

\[ t = 0 \quad \text{c} \quad \text{\( \rightarrow \)} \quad \text{\( v \)} \quad \text{\( \rightarrow \)} \quad \text{\( v \cdot t_F \)} \]

\[ t = t_F \]

\[ c t_F - v t_F = \frac{L}{2} = \frac{L}{2} \frac{\sqrt{1 - v^2/c^2}}{c^2} \]

\[ t_F = \frac{1}{c-v} \frac{L}{2} \frac{\sqrt{1 - v^2/c^2}}{c^2} \]

This is the time Alice measures between the flash and the front clock starting.

Thus, Alice don't see two Bob's clock starting simultaneously; she measures Bob's back clock starting earlier than his front clock, by:

\[ t_F - t_B = \frac{1}{c-v} \frac{L}{2} \frac{\sqrt{1 - v^2/c^2}}{c^2} - \frac{1}{c+v} \frac{L}{2} \frac{\sqrt{1 - v^2/c^2}}{c^2} = \]

\[ = \left[ \frac{1}{c-v} - \frac{1}{c+v} \right] \frac{L}{2} \frac{\sqrt{1 - v^2/c^2}}{c^2} = \frac{v L}{c^2} \frac{1}{\sqrt{1 - v^2/c^2}} \]
Now if Alice takes simultaneous pictures of Bob's two clocks at the time she sees the first (front) clock to start, what would the back clock read? We assume that two Alice's cameras are positioned exactly next to two clocks.}

In Alice's frame the back clock were running for \[ \Delta t = \frac{v t}{c^2} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \] when the picture is taken.

However, the clocks run in Bob's frame, and his time runs slower by \[ t' = \frac{t}{\gamma} = \frac{v t}{c^2} \]
thus the back clock should show \[ \Delta t' = \Delta t / \gamma = \frac{v^2}{c^2} \].

Possible controversy: in Bob's frame two clocks are perfectly synchronized, so why two pictures show different times?
That is because Alice's cameras are synchronized only in her ref. frame.

Following the same discussion we can find that Bob sees two pictures taken at different times: the camera \( P_F \) goes off first at \( t' = 0 \), and then \( \Delta t' = \frac{v^2}{c^2} \) the second one \( P_B \) takes the picture of the back clock.
We will explore the idea of relative simultaneity and relative time dilation / space contraction in the following example.

Suppose we have two stationary clocks $C_1, C_2$ separated by 6 light seconds $l = 6c = 18 \cdot 10^8$ m. The third clock $C'$ are installed on a space shuttle travelling at $v = 0.6c$ along the line at which two stationary clocks are mounted. $C_1$ and $C'$ start at the same time when the shuttle passes $C_1$.

Bob: $C'$

Alice: $C_1$ -- $C_2$

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**Question #1:** What would $C_2$ and $C'$ read when the shuttle passes $C_2$?

Clock $C_2$: Since the shuttle travels at $v = 0.6c$, it takes it $\frac{6c}{0.6c} = 10s$ to reach $C_2$, so $C_2$ read 10s.

Clock $C'$: Time runs slower in a moving frame by a factor of $\frac{1}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}} = 0.8$, so $C'$ should read $0.8 \cdot 10s = 8s$.

Alternatively, the distance b/w $C_1$ and $C_2$ seems shorter, so it will take $\Delta t' = \frac{0.8l}{v} = 8s$ for the shuttle to reach $C_2$. 
So if Alice takes simultaneous pictures with local cameras of all three clocks, she would see

C (08:00) →

C [10:00] → C [10:00]

Question #2

However, Alice moves with respect to Bob, so he should observe her time to be dilated as well!

To resolve this we have to remember that from Bob's point of view, Alice's clocks are not synchronized!

Namely, C₂ should be ahead of C₁ by \( \frac{V \cdot \Delta \ell}{c^2} = 3.6s \)

Thus, at the time the shuttle passes C₁, the clock C₂ must show 3.6s if Bob makes the simultaneous measurements.

Similarly, when Bob reaches C₂ (and see 10s reading), the first clocks should read 10s - 3.6s = 6.4s.

Bob's instant pictures (using local cameras)

Thus, Bob decides that it took him only 6.4s = 0.88 sec from Alice's point of view to travel from C₁ to C₂, and her time is dilated.
We can use Lorentz transformations to find what different clocks are showing.

- Start position
- \( t_1' = 0 \) for Bob in \( C_1 \)
- \( x_1' = 0 \)

- \( t_1 = 0 \) for Alice in \( C_1 \)
- \( x_1 = 0 \)

When asked what time \( C_2 \) shows at the same time, Alice and Bob would disagree (since \( C_2 \) is a distant clock).

\[ x_2 = 6.6 \]

Alice: \( C_1 \) and \( C_2 \) are synchronized, thus \( t_1 = t_2 = 0 \).

We can use Lorentz transformation to see where and when Bob observed this "observation" of the second clock by Alice.

Coordinates for observations in Alice's RF:

\[ \begin{cases} x_2 = 6.6 \\ t_2 = 0 \end{cases} \]

\[ x_2' = \gamma (x_2 - v t_2) = \frac{5}{4} \cdot 6.6 = 7.5 \text{ c} \]

\[ t_2' = \gamma (t_2 - x_2 / c^2) = -\frac{5}{4} \cdot \frac{6.6}{c^2} = -4.5 \]

According to Bob, Alice has checked her second clock before she checked her first.

Suppose Bob checks what time \( C_2 \) shows when \( C_1 \) shows \( t_1 = 0 \) (simultaneous measurement in his RF):

\[ t_2' = \frac{x_2' v}{c^2} = 0 \quad \text{and} \quad (t_1, t_2) = 0 \]

\[ t_2 = x_2' v / c^2 = 3.6 \text{ s} \]
Finish point

\[ \begin{align*}
& \text{Event } E_1, \text{ time } t_1^1 = 8s \\
& \text{Event } E_2, \text{ time } t_2 = 10s
\end{align*} \]

What time \( E_1 \) shows "at the same time" as \( E_1 \) overtakes \( E_2 \)?

Again, Alice and Bob will have different answers because their "at the same time"s differ.

For Alice \( t_1 = t_2 = 10s \) (since both clocks are stationary in her RF)

For Bob: if \( t_1^1 = t_2^1 = 8s \), what time \( \tilde{t}_1 \) corresponds to in Alice's RF?

\[ t_1^1 = 8(\tilde{t}_1 - \frac{v}{c^2} x_1) = \frac{\tilde{t}_1}{c} \Rightarrow \tilde{t}_1 = t_1(t_1/8) = 6.4s \]
Question #3
If Bob does not have a camera positioned to take simultaneous (in his frame) pictures of C₁ and C₂, can he still measure what the reading of C₁ was when he passed C₂?
Suppose both Alice and Bob take pictures of C₁ at the moment C₁ passes C₂ (so they synchronize the timing for the pictures).
Since both cameras are located in C₂, and they are taking pictures of a distant object (C₁), we must take into account the time it takes to travel from C₁ to C₂.
Alice's rest frame: Since it takes light 6 s to go from C₁ to C₂, the photo must show C₁ 6 seconds ago, i.e., reading 4 s.
Bob's picture must be identical, since both cameras were at the same place in space and time, and captured the same physical reality.
But is that what Bob expects to see?
Naive expectation #1

Since \( c' \) measures 8s, and it takes 6s for light to reach \( C_2 \), the photo should show 2s.

Wrong because light travels a shorter distance in Bob's frame.

Naive expectation #2

It takes \( \frac{0.8l}{c} = 4.8s \) to reach \( C_2 \) in Bob's frame.

Wrong - in Bob's frame the clocks move, so light travels shorter distance!

Time between the light is emitted and then captured in a photograph \( \Delta t' \)

\[ c \Delta t' + v \Delta t' = l' \quad \Rightarrow \quad \Delta t' = \frac{l'}{c + v} = \frac{0.8 \cdot 6c}{1.6c} = 3s \]

Naive expectation #3

Triptime = 8s, and light travels 3s, so \( C_1 \) reads 8s - 3s = 5s.

Wrong - 8s and 3s times are measured in Bob's frame, and \( C_1 \) shows time in Alice's frame, moving with respect to Bob, so her time is dilated.
It is correct that in Bob's frame it takes light from $C_1$, 3s to reach the camera, so the shuttle travels 5s since $C_1$ and $C_i$ were started at zero at the same time.

But in Alice's (moving) frame time is dilated, so only $0.8 \cdot 5s = 4s$ has passed, so $C_i$ shows 4s when Bob takes picture of $C_1$ 8s after passing it, and (in his frame) light from $C_1$ travels 3s, and thus shows $C_1$ as it looked 3s before, the reading on $C_1$ is \(04:00\).

Similarly, since in Bob's frame it takes light only 3s to reach the camera, in Alice's frame only $0.8 \cdot 3s$ has passed. So if the photograph shows 8s reading, then at the moment the photograph is taken the clock $C_i$ must read $4s + 2.4s = 6.4s$, as expected.