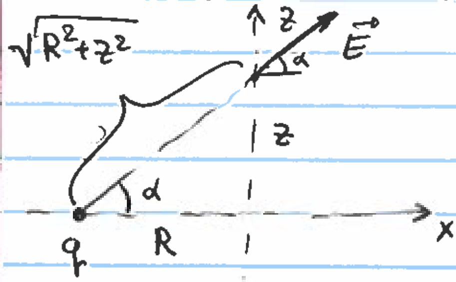
Electric field and potential of continuous charge distributions

Point charge



$$|\vec{E}| = \frac{kq}{(R^2+z^2)}$$

$$E_x = |\vec{E}|\cos\alpha = kq\frac{R}{(R^2+z^2)^{3/2}}$$

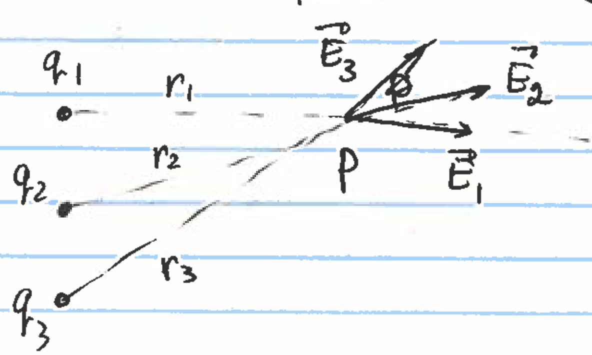
$$E_z = |\vec{E}|\sin\alpha = kq\frac{z}{(R^2+z^2)^{3/2}}$$

$\cos\alpha = R/\sqrt{R^2+z^2}$

$\sin\alpha = z/\sqrt{R^2+z^2}$

$V = \frac{kq}{\sqrt{R^2+z^2}}$ scalar (no direction)

Multiple point charges



$$\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$$

$$E_{tot\,x} = E_{1x} + E_{2x} + E_{3x}$$

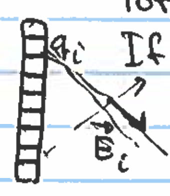
(same for all components)

$$V_{tot} = V_1 + V_2 + V_3 =$$

$$= \frac{kq_1}{r_1} + \frac{kq_2}{r_2} + \frac{kq_3}{r_3}$$

Continuous line of charges

Total charge $-Q$



If breaking into N "pieces" $\longrightarrow \Delta q_i = Q/N$

$$\vec{E}_{tot} = \sum_{i=1}^{N} \frac{k\Delta q_i}{r_i^2}\hat{r}_i$$

If a size of each piece is Δx $\quad N = \frac{L}{\Delta x}$

$$\Delta q_i = \frac{Q}{L}\Delta x$$

$\Delta x \rightarrow dx$

$\Delta q \rightarrow dq$

$$dq = \frac{Q}{L}dx$$

Charge density

1D

Uniformally distributed charge Q

Linear charge density

$$\lambda = \frac{Q}{L} = \frac{\text{total charge}}{\text{total length}}$$

(lambda)

$dq = \lambda\, dx$

2D

total area A

Uniform charge distribution.

surface density

$$\sigma = \frac{\text{total charge}}{\text{total area}} = \frac{Q}{A}$$

(sigma)

$dq = \sigma\, dx\, dy$ or $dq = \sigma \cdot r\, dr\, d\theta$ (polar coordinates)

square of side d $\quad \sigma_{sq} = \frac{Q}{d^2}$

disc of radius R $\quad \sigma_{disc} = \frac{Q}{\pi R^2}$

3D Volume charge density

$$\rho = \frac{\text{total charge}}{\text{total volume}} = \frac{Q}{V}$$

(rho)

Cube with side d $\quad \rho_{cube} = \frac{Q}{d^3}$

Sphere of radius R $\quad \rho_{sphere} = \frac{Q}{\frac{4\pi}{3} R^3} = \frac{3Q}{4\pi R^3}$

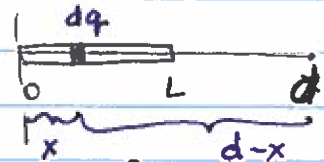
$dq = \rho\, dx\, dy\, dz$

or

$dq = \rho \cdot r^2 \sin\theta\, dr\, d\theta\, d\varphi$ spherical coordinates

Examples of calculations

1D case



$\lambda = Q/L$

Position of an elementary charge along the rod

$x = [0, L]$

$dq = \lambda\, dx$

distance to the point of interest : $d-x$

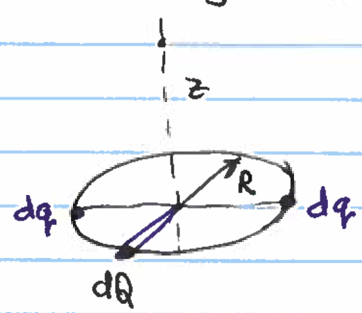
$$dE_x = \frac{k\,dq}{(d-x)^2} = \frac{k\lambda\, dx}{(d-x)^2}$$

$$E_x = \int_{x=0}^{L} dE_x = \int_{x=0}^{L} (k\lambda) \frac{dx}{(d-x)^2} = (k\lambda) \left.\frac{1}{d-x}\right|_0^L =$$

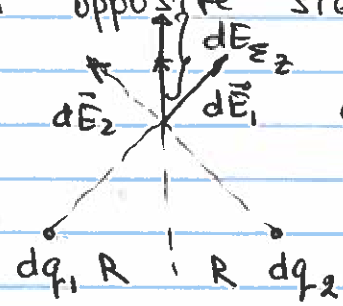
$$= k\lambda \left(\frac{1}{d-L} - \frac{1}{d} \right) = k\underbrace{\lambda \frac{L}{d(d-L)}}_{=Q} = \frac{kQ}{d(d-L)}$$

2D case

Uniformly charged ring



Convenient to pick two elementary charges on opposite sides

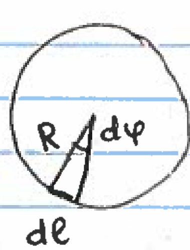


Only vertical component

Charge density (linear)

$$\lambda = \frac{Q}{2\pi R}$$

$dq = \lambda\, dl = \lambda R\, d\varphi$

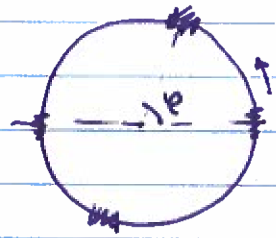


For each pair

$$dE_z = 2\,dE_z = \frac{2k\,dq}{(z^2+R^2)^{3/2}} \cdot z$$

$$= \frac{2k\lambda R \cdot z}{(z^2+R^2)^{3/2}}\, d\varphi$$

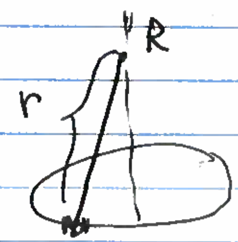
Each pair of oppositely located elementary charges will produce identical contributions!



$$E_{tot} = \int_{\varphi=0}^{\pi} dE_z = \frac{2k\lambda R z}{(z^2+R^2)^{3/2}} \int_0^{\pi} d\varphi =$$

$$= \frac{2k\lambda R z\,\pi}{(z^2+R^2)^{3/2}} = kQ \frac{z}{(z^2+R^2)^{3/2}}$$

Electric potential calculation is even simpler



Every elementary charge dq is at the same distance from P

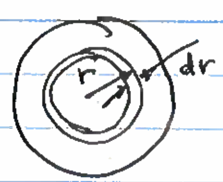
$$r = \sqrt{R^2+z^2}$$

$$dV = \frac{k\,dq}{r} \qquad V = \frac{k}{r}\int_{ring} dq = \frac{kQ}{r}$$

$$V_{ring} = \frac{kQ}{(z^2+R^2)^{1/2}}$$

Uniformally charged disc

$$\sigma = \frac{Q}{\pi R^2}$$



Let's break it into "elementary rings"

Charge of an individual ring

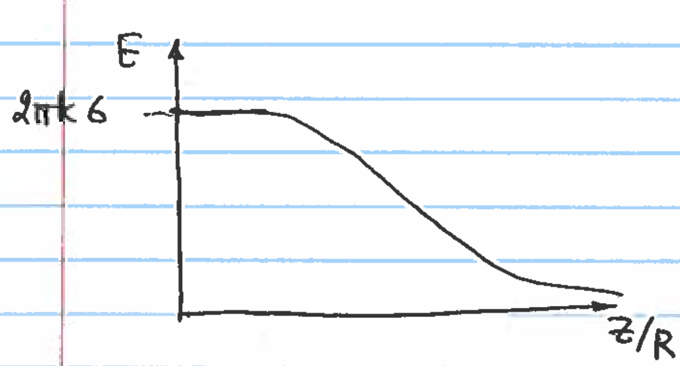
$$dQ(r) = \sigma \cdot 2\pi r\,dr$$

$$dE_{ring} = k\,dQ(r) \cdot \frac{z}{(z^2+r^2)^{3/2}} = 2\pi\sigma k \frac{z r\,dr}{(z^2+r^2)^{3/2}}$$

$$E_{ring} = \int_{r=0}^{R} dE_{ring} = 2\pi\sigma k \int_0^R \frac{z r\,dr}{(z^2+r^2)^{3/2}}$$

$$E_{ring} = 2\pi\sigma k z \int_0^R \frac{r\,dr}{(z^2+r^2)^{3/2}} = 2\pi\sigma k z \left[-\frac{1}{\sqrt{z^2+r^2}}\right]\Bigg|_0^R$$

$$= 2\pi\sigma k z \left[\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right] = 2\pi k\sigma \left[1 - \frac{z}{\sqrt{z^2+R^2}}\right]$$



If $z \ll R$

$$E = 2\pi k\sigma = \frac{2\pi\sigma}{4\pi\varepsilon_0} = \frac{\sigma}{2\varepsilon_0}$$

If $z \ll R$, we can neglect edges, and get the electric field of an infinite plane.

Purely for fun ! if $z \gg R$

$$\frac{1}{\sqrt{z^2+R^2}} = \frac{1}{z}\frac{1}{(1+R^2/z^2)^{1/2}} \approx \frac{1}{z}\left(1 - \frac{1}{2}\frac{R^2}{z^2}\right)$$

$$\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}} \approx \frac{R^2}{2z^3}$$

$$\underset{z \gg R}{E_{ring}} \approx \cancel{2\pi}\,\frac{Q}{\cancel{\pi R^2}} \cdot k \cdot \cancel{z} \cdot \frac{\cancel{R^2}}{\cancel{2}z^3} = \frac{kQ}{z^2}$$

From afar a disc looks like a point charge