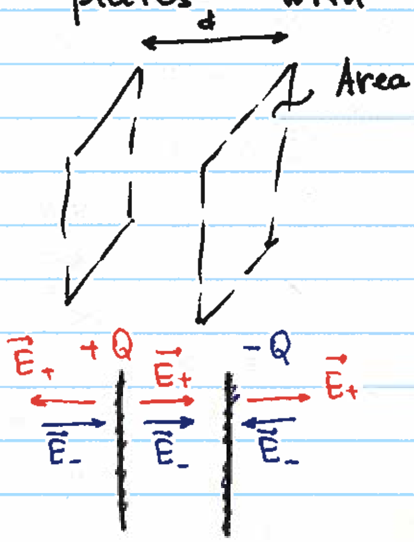
Capacitors

Capacitors are electric components designed to store electric energy

Classical capacitor consits of two parallel plates with equal and opposite charge



Single plate with surface charge density $\sigma = Q/A$

$$\vec{E}_{plate} = \frac{\sigma}{2\varepsilon_0} = \frac{Q}{2\varepsilon_0 A}$$

Outside of the capacitor

$$E_{tot} = 0$$

Inside the capacitor

$$E_{tot} = 2E_{plate} = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$$

Potential difference (voltage) b/w two plates

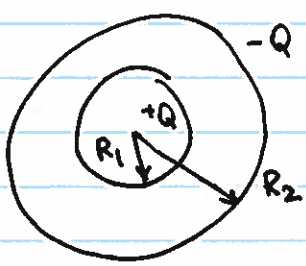
$$\Delta V = V_+ - V_- = E_{tot} \cdot d = Q \cdot d / \varepsilon_0 \cdot A$$

$$\boxed{\Delta V} = \frac{d}{\varepsilon_0 A} \cdot \boxed{Q} = Q/C$$

$$C = \frac{\varepsilon_0 A}{d}$$

capacitence, depends on the geometry of the capacitor, and defines the ratio b/w its charge and voltage

Spherical capacitor



$$V_+ = \cancel{kQ}\ \frac{kQ}{R_1} - \frac{kQ}{R_2}$$

$$V_- = \frac{kQ}{R_2} - \frac{kQ}{R_2} = 0$$

$$V_+ - V_- = \left(\frac{k}{R_1} - \frac{k}{R_2}\right) Q = \underbrace{\frac{1}{4\pi\varepsilon_0}\ \frac{R_2 - R_1}{R_2 \cdot R_1}}_{1/C_{sph}} \cdot Q$$

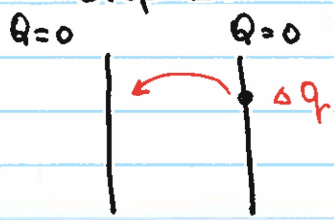
Capacitence of a spherical capacitor

$$C_{sph} = 4\pi\varepsilon_0\ \frac{R_2 R_1}{R_2 - R_1}$$

Energy stored in a capacitor

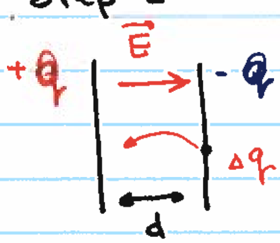
Charging process

Step 1: uncharged capacitor



It takes no energy to move the first charge, as there is no electric field

Step 2



Work that needs to be done to move Δq from $-Q$ to $+Q$ is $\text{Work} = \Delta q \cdot E \cdot d = \Delta q \cdot \boxed{(V_+ - V_-)}$

standard label $-V$

$$= \frac{q}{C}\,\Delta q$$

Total work to charge ~~the~~ a capacitor from uncharged to charge Q

$$\text{Work}_{total} = \oint \sum_{\Delta q = 0}^{Q} \text{work}_{\Delta q} = \int_0^Q \frac{q}{C}\,dq = \frac{Q^2}{2C}$$

Equal to energy stored in a capacitor

Depending on what we know it may be convenient to use various versions of the energy expression since $Q = V \cdot C$

$$U_{capacitor} = \frac{Q^2}{2C} = \frac{1}{2} V^2 C = \frac{QV}{2}$$

We can also easily find the energy density (how much energy is stored per unit volume)

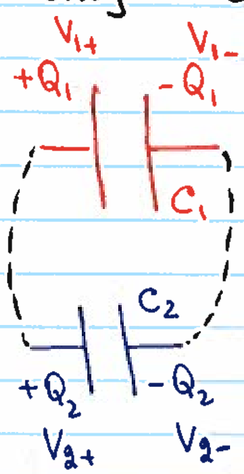
$$\mathcal{V} = A \cdot d$$

$$U = \frac{Q^2}{2C} \frac{1}{\mathcal{V}} = \frac{Q^2 \cdot \cancel{d}}{2 \cdot \varepsilon_0 \cdot A} \frac{1}{A \cdot \cancel{d}} = \frac{1}{2} \underbrace{\left[\frac{Q}{\varepsilon_0 A}\right]^2}_{E^2} \cdot \varepsilon_0 = \frac{1}{2} \varepsilon_0 E^2$$

This is a universal definition of the energy density of the electric field.

Capacitor connections

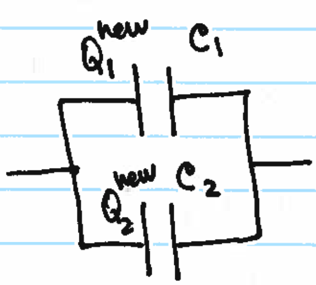
In the equilibrium all points of a conductor have the same electric potential (i.e. internal electric field is zero).
Thus, in equilibrium the voltage across any conductor (wire) is zero



What happens when we connect two charged capacitors?
We need to know their potentials to say!

If $V_{1+} \neq V_{2+}$ (and $V_{1-} \neq V_{2-}$), the charge will flow b/w them through the wire until the potentials are the same

$\Downarrow$

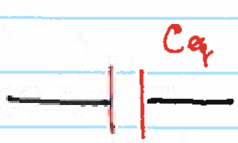


$V = V_+ - V_-$ — same for all capacitors

$$Q_1^{new} = C_1 \cdot V \qquad Q_2^{new} = C_2 V$$

$$Q_{tot} = \underbrace{Q_1^{new} + Q_2^{new}}_{\text{after the connection}} = \underbrace{Q_1 + Q_2}_{\text{before the connection}}$$

Equivalent single capacitor

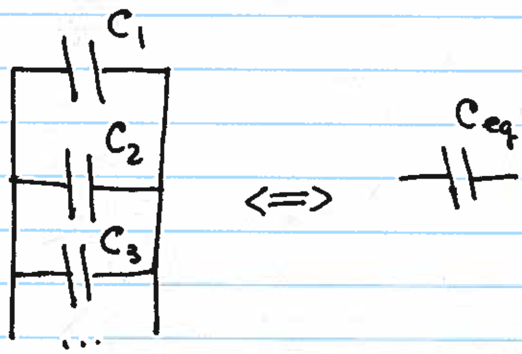


$$Q_{eq} = Q_{tot} = C_1 \cdot V + C_2 \cdot V = (C_1 + C_2) \cdot V$$

equivalent capacitance $C_{eq} = \dfrac{Q_{eq}}{V}$

$$C_{eq} = C_1 + C_2$$

For many capacitors connected in parallel

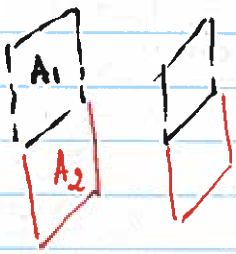


$$V_1 = V_2 = V_3 = V_{eq}$$

$$Q_1 + Q_2 + Q_3 + \ldots = Q_{eq}$$

$$C_1 + C_2 + C_3 + \ldots = C_{eq} = \frac{Q_{eq}}{V_{eq}}$$

Parallel connection is equivalent to "stacking" capacitor plates



$$C_{eff} = \frac{\varepsilon_0 (A_1 + A_2)}{d} = \frac{\varepsilon_0 A_1}{d} + \frac{\varepsilon_0 A_2}{d} = C_1 + C$$