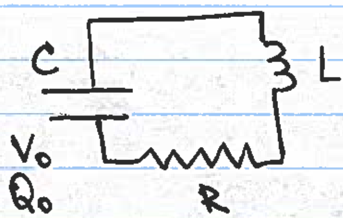
AC circuits — with complex numbers!

Let's get back to RLC circuit



$$\frac{Q}{C} + IR + L\frac{dI}{dt} = 0 \qquad I = -\frac{dQ}{dt}$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

Hard to calculate : if $Q(t) = Q_0 \underline{\cos\omega t}$

$$\frac{dQ}{dt} = -Q_0\omega \underline{\sin\omega t}$$

doable with a lot of trig identities ⌢̈

Let's for a moment consider overdamped ~~sus~~ circuit (large R) → no oscillations

so we can guess $Q(t) = Q_0 e^{-\beta t}$ (decay)

$$L\beta^2 \cancel{(Q_0 e^{-\beta t})} - R\beta\cancel{(Q_0 e^{-\beta t})} + \frac{1}{C}\cancel{(Q_0 e^{-\beta t})} = 0$$

$$L\beta^2 - R\beta + \frac{1}{C} = 0$$

$$\beta_{1,2} = \frac{R \pm \sqrt{R^2 - 4L/C}}{2L}$$

If $R > \sqrt{4L/C}$ β is a real number

(no oscillations, as expected)

But what if $R < \sqrt{4L/C}$? $\Rightarrow \sqrt{R^2 - \frac{4L}{C}} = i\sqrt{\frac{4L}{C} - R^2}$

$$\beta_{1,2} = \underbrace{\frac{R}{2L}}_{\text{real part}} \pm i\underbrace{\sqrt{\frac{1}{LC} - \frac{R^2}{4L}}}_{\text{imaginary part}}$$

if $R = 0$ perfect oscillations $\beta_{1,2} = \pm\frac{i}{\sqrt{LC}} = \pm i\omega_{LC}$ $\quad Q(t) \propto e^{\pm i\omega_{LC} t}$

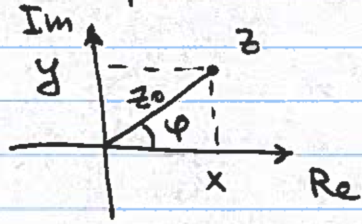
Why physicists love complex numbers

Euler formula $z = e^{i\varphi} = \cos\varphi + i\sin\varphi$
for any $|z| = 1$ complex number

if $|z| = z_0$ $\Rightarrow$ $z = z_0 e^{i\varphi} = z_0 \cos\varphi + i z_0 \sin\varphi$

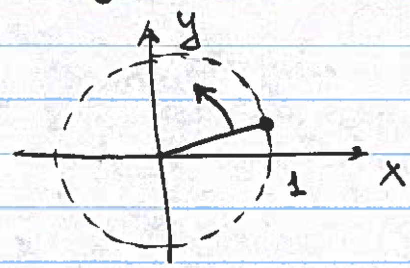
A complex number can represent
a point in 2D $z = x + iy$

$z_0 = \sqrt{x^2+y^2}$ (the length)



$x = z_0 \cos\varphi$
$y = z_0 \sin\varphi$

Imagine a rotating point with frequency ω



$\varphi = \omega t$ if $x(t=0) = 1$, $y(t=0) = 0$

$x(t) = \cos\omega t = \mathrm{Re}\,[e^{i\omega t}]$

$y(t) = \sin\omega t = \mathrm{Im}\,[e^{i\omega t}]$

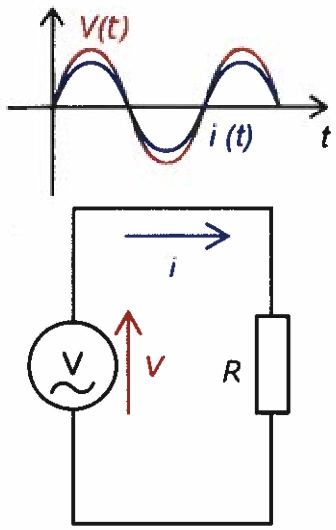
So for many problems with oscillations or wave, it is convenient use complex exponents instead of cos / sine functions, and just take real part at the end

Alternatively, one can use

$$\cos\varphi = \frac{1}{2}\left(e^{i\varphi} + e^{-i\varphi}\right)$$

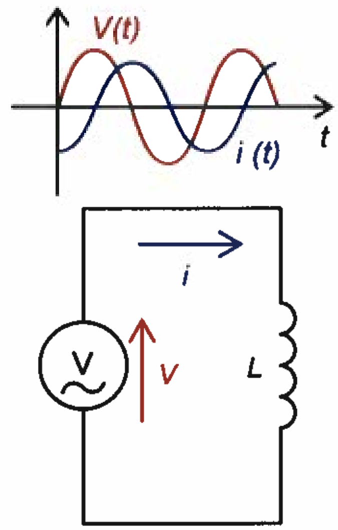
$$\sin\varphi = \frac{1}{2i}\left(e^{i\varphi} - e^{i\varphi}\right)$$

Voltage and current are in phase



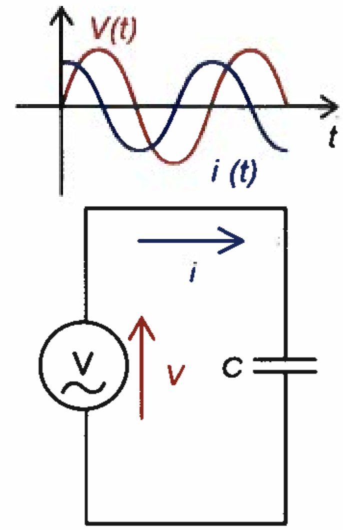
$$i(t) = \frac{V_{max}}{R} \sin \omega t$$

The current lags behind the voltage by a phase of 90°



$$i(t) = \frac{V_{max}}{\omega L} \sin\left(\omega t - \frac{\pi}{2}\right)$$

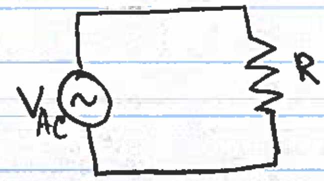
The current leads the voltage by a phase of 90°



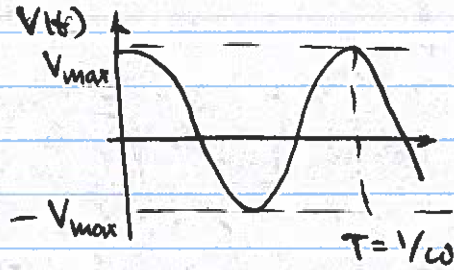
$$i(t) = \omega C V_{max} \sin\left(\omega t + \frac{\pi}{2}\right)$$

AC circuit → with oscillating voltage source

①



$$V_{AC}(t) = V_{max}\cos\omega t$$



If such source is connected to a resistor, what is the current through a resistor?

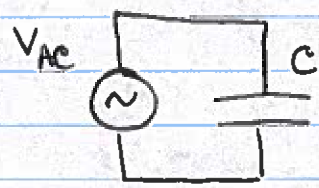
$i_R(t)$ - ?

$$V_{AC}(t) - i_R \cdot R = 0$$

$$i_R = \frac{V_{AC}(t)}{R} = \frac{V_{max}}{R}\cos\omega t$$

in phase with the voltage source

②



$$V_{AC} - \frac{Q(t)}{C} = 0$$

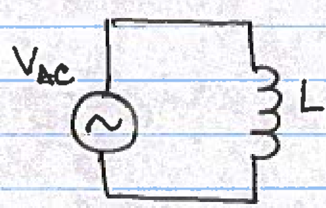
$$Q(t) = C \cdot V_{AC}(t) = C \cdot V_{max}\cos\omega t$$

$V_{AC} = V_{max}\cos\omega t$

$$I(t) = \frac{dQ}{dt} = -CV_{max}\omega\sin\omega t$$

$$I(t) = -(C\omega)\cdot V_{max}\sin\omega t$$

③



$$V_{AC}(t) - L\frac{dI}{dt} = 0$$

$$\frac{dI}{dt} = \frac{1}{L}V_{AC}(t) = \frac{V_{max}}{L}\cos\omega t$$

$$I(t) = \frac{V_{max}}{L\omega}\sin\omega t$$

$V_{AC}(t) = V_{max} \cos\omega t \longrightarrow I_R = \frac{1}{R} V_{max} \cos\omega t$

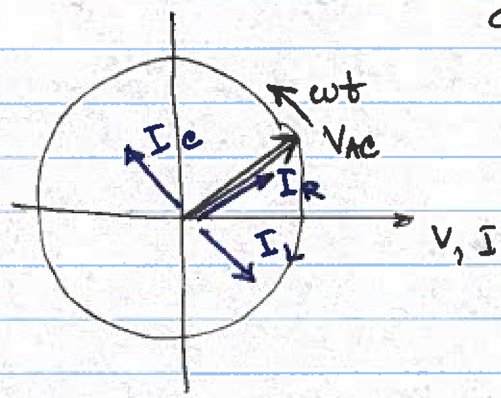
same phase

$\longrightarrow I_c = \omega C \cdot V_{max} \cos(\omega t + \frac{\pi}{2})$

a quater cycle ahead of V_{AC}
or $\pi/2$ aread

$\longrightarrow I_L = \frac{1}{\omega L} V_{max} \cos(\omega t - \frac{\pi}{2})$

a quater cycle $(\pi/2)$ behind



If $V_{AC} = Re[V_{max} e^{i\omega t}]$

then $I_R = Re[\frac{V_{max}}{R} e^{i\omega t}]$

$I_L = Re[\frac{V_{max}}{\omega L} e^{i\omega t} \cdot e^{-i\pi/2}]$

$I_c = Re[\omega C\, V_{max} e^{i\omega t} e^{i\pi/2}]$

$e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = i$

$e^{-i\pi/2} = \cos\pi/2 - i\sin\pi/2 = -i = \frac{1}{i}$

$I_R = Re[\cancel{\frac{V_{max}}{R}}\ \frac{1}{R} \cdot V(t)]$

$I_c = Re[\underbrace{i\omega C}_{1/Z_c} \cdot V(t)]$ $I_L = Re[\underbrace{\frac{1}{i\omega L}}_{1/Z_L} V_{AC}$

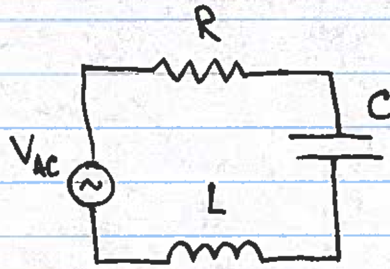
reactance $\longrightarrow$ similar to resistance, and behaves the same way

$Z_c = \frac{1}{i\omega C}$ $Z_L = i\omega L$

Reactance Z behave just as normal resistance R in Ohm's law

$$V_R = R \cdot I \qquad V_L = Z_L \cdot I = i\omega L \cdot I \qquad V_C = Z_C \cdot I = \frac{1}{i\omega C} I$$

$V_{AC}(t) = V_m \cos\omega t = Re\left[V_m e^{i\omega t}\right] \longrightarrow$ do calculations with $V_{ac}(t) = V_m e^{i\omega t}$



$$V_{AC} - IR - \frac{Q}{C} - L\frac{dI}{dt} = 0$$

vs

$$V_{AC} - R \cdot I - Z_C \cdot I - Z_L \cdot I = 0$$

$$I = \frac{V_{AC}}{R + Z_C + Z_L} = \frac{V_m}{R + i\omega L + \frac{1}{i\omega C}} e^{i\omega t}$$

To calculate real current we need to take real part of I now

$$I(t) = Re\left[\underbrace{\frac{V_m}{R + i\omega L + 1/i\omega C}}_{\text{convert into polar form}} e^{i\omega t}\right]$$

$$x + iy = \sqrt{x^2+y^2}\,(\cos\varphi + i\sin\varphi) = \sqrt{x^2+y^2}\left[\underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{\cos\varphi} + i\,\underbrace{\frac{y}{\sqrt{x^2+y^2}}}_{\sin\varphi}\right]$$

$$\tan\varphi = x/y$$

$$R + i\omega L + \frac{1}{i\omega C} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\; e^{i\varphi_{RLC} \cdot t} \qquad \varphi_{RLC} = \tan^{-1}\frac{R}{\omega L - \frac{1}{\omega C}}$$

$$I(t) = Re\left[\frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{-i\varphi_{RLC} \cdot t}\, e^{i\omega t}\right]$$

$$I(t) = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos(\omega t - \varphi_{RLC})$$

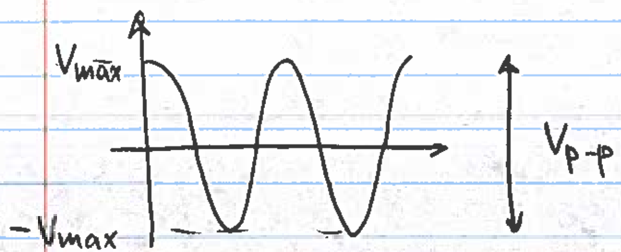
$$I_m = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

maximum amplitude

resonance @ $\omega L = \frac{1}{\omega C}$

$$\omega^2 = 1/LC = \omega_{LC}^2$$

AC circuits terminology



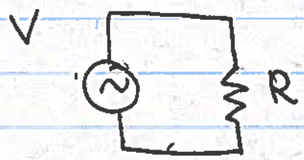
$V_{AC}(t) = V_{max} \cos \omega t$

V_{max} – amplitude

Peak-to-peak $V_{p-p} = 2V_{max}$

What is V_{RMS}?

rms → root mean square

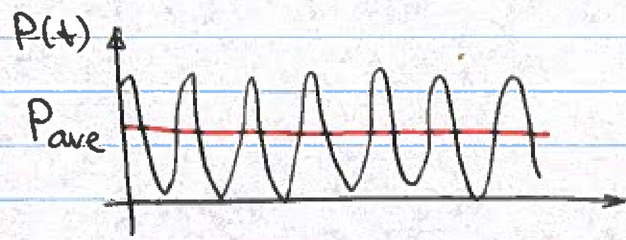


$V(t) = V_{max} \cos\omega t$

$I(t) = V_{max}/R \cos\omega t$

Instantaneous power $P(t) = V \cdot I =$

$$= \frac{V_{max}^2}{R} \cos^2\omega t$$



When averaged over many oscillations

$$\langle \cos^2 \omega t \rangle = \frac{1}{2}$$

$$P_{ave} = \frac{1}{2} \frac{V_{max}^2}{R} = \frac{(V_{max}/\sqrt{2})^2}{R}$$

If we have a circuit with constant voltage source $\mathcal{E} = V_{max}/\sqrt{2}$, it will put out the same power over the long time as ac circuit with amplitude V_{max}

$$V_{RMS} = \frac{V_{max}}{\sqrt{2}} \qquad I_{RMS} = \frac{I_{max}}{\sqrt{2}}$$