DC circuits with capacitors

- If a circuit consists of batteries and resistors only, all changes are (nearly) instantaneous, and values of currents & voltages does not depend on time.

It's not the case when capacitors are involved!

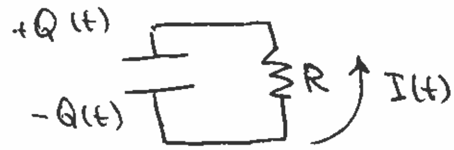
Example 1: charged capacitor connected to a resistor

Q_{max} – original charge

t = 0

as time flows the charge on the capacitor reduces, and energy gets lost to heat on the resistor

$$\boxed{I_{max} = \frac{V_{max}}{R} = \frac{Q_{max}}{CR}}$$



After the long time all charges are zero, no current

- $V_{capacitor} = V_{resistor}$ at any time

$$\frac{Q(t)}{C} = -I(t)R \qquad \text{but also} \quad I(t) = \frac{dQ(t)}{dt}$$

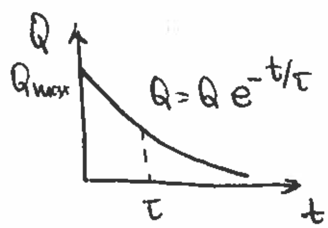
$$\frac{dQ(t)}{dt} = -\frac{1}{RC}\,Q(t)$$

Simple(st) differential equation

$$\frac{dy}{dt} = -\frac{1}{\tau}\cdot y \Rightarrow \underline{y(t) = y(0)\,e^{-t/\tau}}$$

$$Q(t) = Q_{max}\,e^{-t/RC}$$

$\tau = RC$ is a time constant of an RC circuit

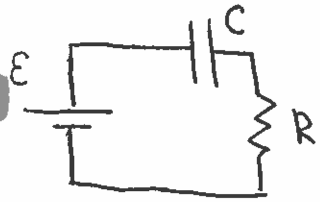


$$I(t) = \left|\frac{dQ}{dt}\right| = \left|-\frac{Q_{max}}{RC}\right| e^{-t/RC} = I_{max}\,e^{-t/RC}$$

$\tau = RC$ is a characteristic time for the capacitor to get discharged.

- So if you need to measure a voltage across a charged capacitor, you must ensure your voltmeter has high resistance!

Let's consider charging a capacitor

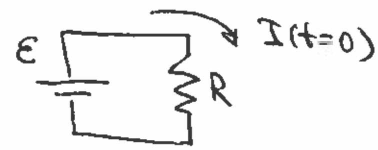


$Q(t=0)=0 \qquad V_C(t=0)=0$

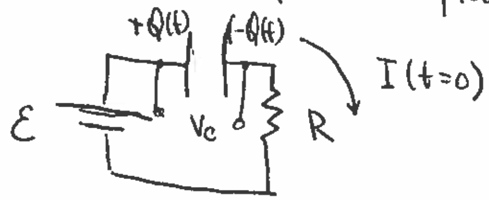
at $t=0$: $\quad \mathcal{E} - I(t=0)\cdot R = 0 \qquad I_{t=0} = \dfrac{\mathcal{E}}{R}$

as if there is no capacitor

Uncharged capacitor is equivalent to a straight wire!



As current flows, it start "depositing" charges on the capacitor plates



$$V_C(t) = -\,Q(t)/C$$

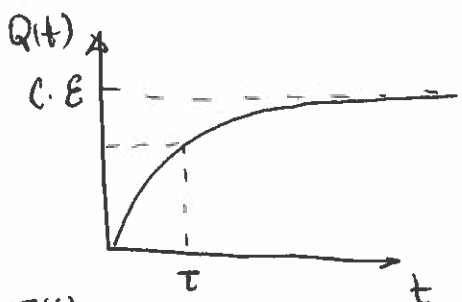
$$\mathcal{E} - Q(t)/C - IR = 0$$

$$\mathcal{E} - dQ/C - \frac{dQ}{dt}\cdot R = 0$$

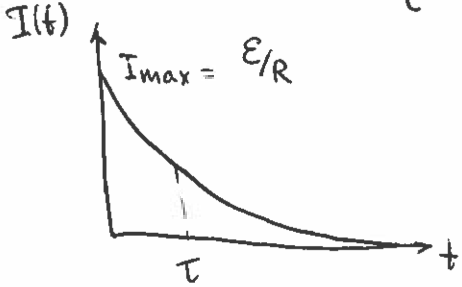
$$\frac{dQ}{dt} = -\frac{Q}{RC} + \frac{\mathcal{E}}{R} = -\frac{(Q - C\mathcal{E})}{RC}$$

$$\int_{Q(t=0)}^{Q(t)} \frac{dQ}{Q - C\mathcal{E}} = -\int_0^t \frac{dt}{RC}$$

$$\ln(Q(t) - C\mathcal{E})\Big|_0^{Q(t)} = \ln\left(\frac{C\mathcal{E} - Q(t)}{C\mathcal{E}}\right) = -\frac{t}{RC}$$



$$\frac{C\mathcal{E} - Q(t)}{C\mathcal{E}} = e^{-t/RC}$$



$$Q(t) = C\cdot\mathcal{E} - C\mathcal{E}\cdot e^{-t/RC} =$$

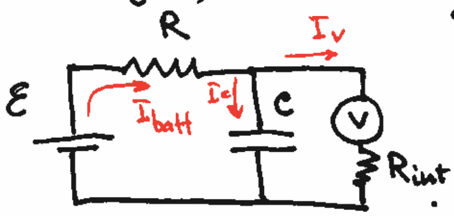
$$= C\mathcal{E}(1 - e^{-t/RC})$$

$$I(t) = \frac{dQ(t)}{dt} = C\mathcal{E}\left(-\left(-\frac{1}{RC}\right)e^{-t/RC}\right) =$$

$$= \frac{\mathcal{E}}{R} e^{-t/RC}$$

So after a long time, when capacitor is charged, it is equivalent to the circuit break.

What if we want to monitor the voltage across the capacitor using a voltmeter?
Ideal voltmeter has infinitely high resistance, but in reality it is just large $R_{int} \gg R$.
Thus, there will be some current flowing through the meter even when the capacitor is fully charged, distorting the ~~de~~ measurement



The voltmeter will show the voltage drop across the capacitor

$I_{batt} = I_c + I_V$ at any moment of time

$$\frac{Q}{C} = I_V \cdot R_{int}$$

$$\mathcal{E} - I_{batt} R - I_V \cdot R_{int} = 0$$

$$\left.\begin{aligned} I_c &= \frac{dQ}{dt} \\ I_V &= Q/R_{int} \cdot C \end{aligned}\right\}$$

$$\mathcal{E} - (I_c + I_V) \cdot R - I_V R_{int} = 0$$

$$\mathcal{E} - I_c \cdot R - I_V \cdot (R + R_{int}) = 0$$

$$\mathcal{E} - I_c \cdot R - \frac{Q}{R_{int} \cdot C}(R + R_{int}) = 0$$

$$\mathcal{E} - \frac{Q}{C}\Big(1 + \underbrace{\frac{R}{R_{int}}}_{\text{correction due to measurements}}\Big) - R\frac{dQ}{dt} = 0$$

$$\frac{dQ}{dt} = \frac{\mathcal{E}}{R} - \frac{Q}{C}\underbrace{\left(\frac{1}{R} + \frac{1}{R_{int}}\right)}_{= 1/R_{eq}} = \frac{\mathcal{E}}{R} - \frac{Q}{C R_{eq}} \Rightarrow \frac{dQ}{\frac{\mathcal{E}}{R} - \frac{Q}{C R_{eq}}} = dt$$

$$\ln\left(\frac{\mathcal{E}/R - Q(t)/C \cdot R_{eq}}{\mathcal{E}/R}\right) = -\frac{t}{C R_{eq}} \qquad R_{eq} = \frac{R \cdot R_{int}}{R + R_{int}}$$

$$Q(t) = \mathcal{E} C \frac{R_{eq}}{R}\left(1 - e^{-t/C R_{eq}}\right)$$

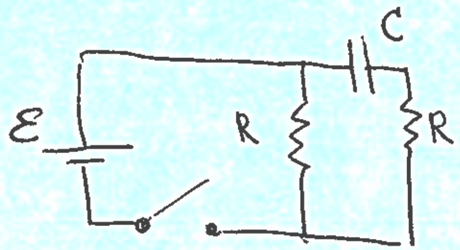
R_{eq} approaches R if $R \ll R_{int}$, but $R_{eq} < R$

when $t \to \infty$ the full capacitor charge is $Q_{final} = \mathcal{E} C \frac{R_{eq}}{R}$

and the measured voltage $V_{final} = Q_{final} \cdot C = \mathcal{E} \cdot \frac{R_{eq}}{R}$

So the measured voltage is smaller than $\mathcal{E}$ by a factor of R_{eq}/R or $R_{int}/R+R_{int}$

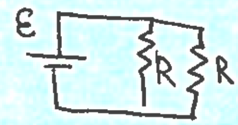
More examples



If the capacitor is not charged and a switch is closed at $t=0$

Immediately after the closing

$t = 0+$



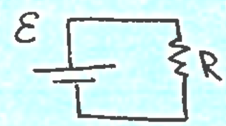
The current through the battery is

$$I = 2\mathcal{E}/R$$

After a very long time

$V_C = \mathcal{E}$, capacitor is fully charged

no current flows through that branch



$$I = \mathcal{E}/R$$

If one needs to calculate how the charge changed in between one would use Kirchhoff's rules (

$$I = I_R + I_C$$

$$\mathcal{E} - I_R \cdot R = 0 \Rightarrow I_R = \mathcal{E}/R$$

$$\boxed{\mathcal{E} - Q(t)/C - I_C R = 0} \qquad I_C = \frac{dQ}{dt}$$

same equation we did already

$$Q_C(t) = \mathcal{E}C\left(1 - e^{-t/RC}\right)$$

$$I_C(t) = \frac{\mathcal{E}}{R} e^{-t/RC}$$

The current through the battery

$$I = I_C + I_R = \frac{\mathcal{E}}{R} + \frac{\mathcal{E}}{R} e^{-t/RC}$$