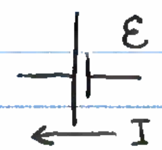
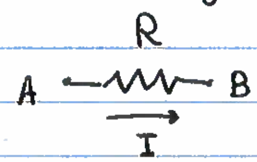
Circuits with capacitors

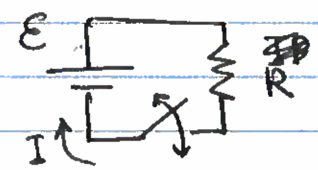
Reminder: a battery provides emf to make electrons move



The voltage drop across the resistor always follows the Ohm's law



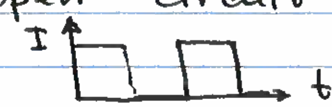
$V_B - V_A = -IR$

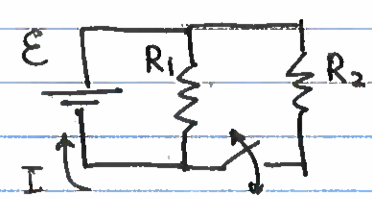


Switch closed $\mathcal{E} = I_c R$

Switch open $I_o = 0$

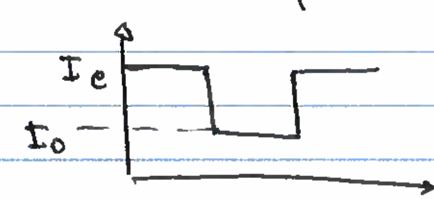
open circuit





Switch closed $\mathcal{E} = I_c \cdot \frac{R_1 R_2}{R_1 + R_2}$

Switch open $\mathcal{E} = I_o \cdot R_1$



Current changes instantaneously and stays ~~on~~ constant

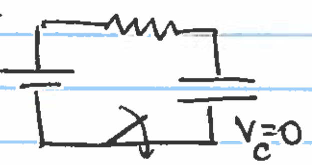
(DC circuits)

Things are different for a capacitor since it charges $\longrightarrow$ its voltage changes

$$V = \frac{Q}{C}$$

no charge $Q=0 \Rightarrow V=0$ (like a wire!)

Example 1:

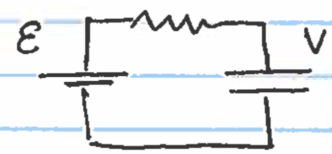


$t=0$ capacitor uncharged

$\xrightarrow{\text{equivalent}}$

$I(t=0) = \mathcal{E}/R$

If we let the capacitor fully charge $t \to \infty$



$\mathcal{E} = V$

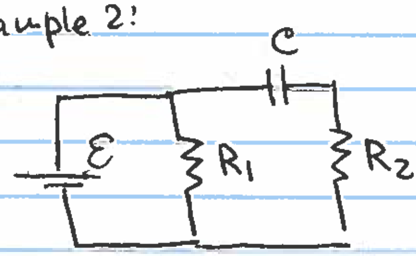
since charges would move until ~~the~~ the voltage across the capacitor matches the emf

$V = \mathcal{E} = Q/C$

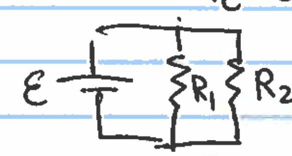
$I(t \to \infty) = 0$

$Q = \mathcal{E} \cdot C$

Example 2:



$t=0$ $V_C = 0$ (= wire)



$\mathcal{E} = I(t=0) \cdot R_{eq}$

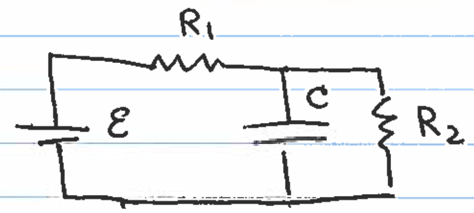
$I(t=0) = \dfrac{\mathcal{E}(R_1+R_2)}{R_1 R_2}$

$t \to \infty$ capacitor is fully charged

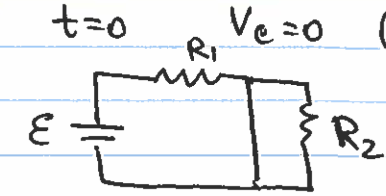
no current through C & R_2

$I(t \to \infty) = \mathcal{E}/R_1$

Example 3:



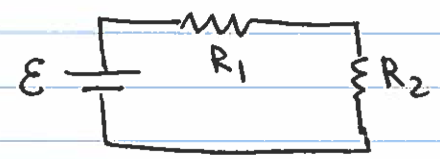
$t=0$ $\quad V_C = 0$ (= wire)



short, no current in R_2

$$I(t=0) = \mathcal{E}/R_1$$

Capacitor is fully charged, no current through C, $V_C = V_{R_2}$



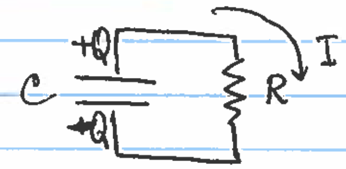
$$I(t\to\infty) = \frac{\mathcal{E}}{R_1+R_2}$$

$$V_C = V_{R_2} = IR_2 = \frac{\mathcal{E}R_2}{R_1+R_2}$$

$$Q = V_C \cdot C = C \cdot \frac{\mathcal{E}R_2}{R_1+R_2}$$

How to calculate $I(t)$?

Simplest example : a charged capacitor C is connected to a resistor R



$$V_C + V_R = 0$$

As time flows the capacitor charge diminishes, its voltage reduces, so the current becomes proportionally smaller

$$V_C = \frac{Q(t)}{C} \qquad V_R = -I(t)R \qquad I(t) = -\frac{dQ}{dt}$$

$$\frac{Q(t)}{C} = I(t)\cdot R = -R\frac{dQ(t)}{dt}$$

$$\frac{dQ(t)}{dt} = -\frac{1}{RC}Q(t)$$

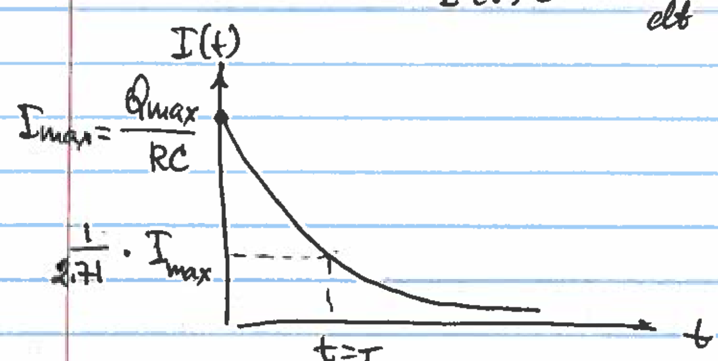
$$Q(t) = Q(t=0)\,e^{-t/\tau} = Q_{max}\,e^{-t/\tau}$$

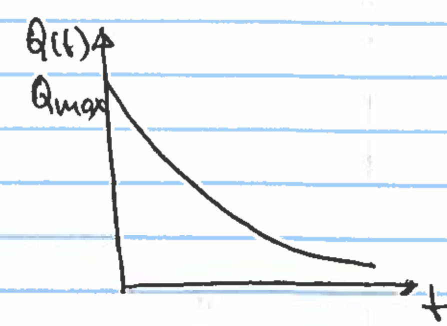
$$\frac{dQ}{dt} = -\frac{1}{\tau}Q_{max}\,e^{-t/\tau}$$

$$-\frac{1}{\tau}Q_{max}e^{-t/\tau} = -\frac{1}{RC}Q_{max}e^{-t/\tau}$$

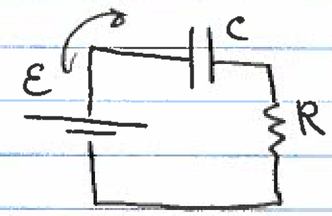
$\boxed{\tau = RC}$ time constant of an RC circuit

$$I(t) = -\frac{dQ}{dt} = \frac{Q_{max}}{RC}\,e^{-t/RC}$$





What about our previous examples?



~~I(t=0)~~ $I(t=0) = \mathcal{E}/R$

$I(t \to \infty) = 0$

Capacitor is charging

$I = \frac{dQ}{dt}$

$$\mathcal{E} - \frac{Q(t)}{C} - IR = 0$$

$$\mathcal{E} - \frac{Q}{C} - R\frac{dQ}{dt} = 0$$

$$\frac{dQ}{dt} = \frac{\mathcal{E}}{R} - \frac{Q}{RC} = \frac{\mathcal{E} \cdot C - Q}{RC}$$

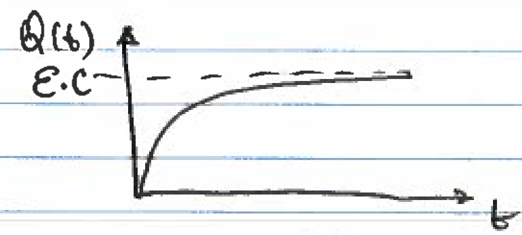
$$\int_{Q(t=0)=0}^{Q(t)} \frac{dQ}{-\mathcal{E}C + Q} = -\int_0^t \frac{dt}{RC}$$

$$\ln(Q - \mathcal{E}C) \Big|_0^{Q(t)} = -\frac{t}{RC}$$

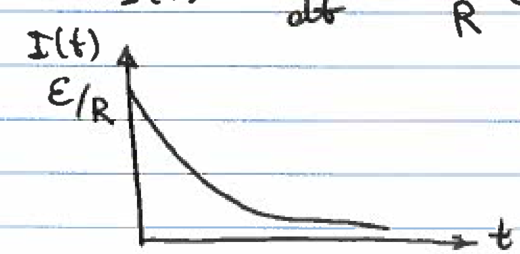
$$\ln\left(\frac{Q - \mathcal{E}C}{-\mathcal{E}C}\right) = \ln\left(\frac{\mathcal{E}C - Q}{\mathcal{E}C}\right) = -\frac{t}{RC}$$

$$\frac{\mathcal{E} \cdot C - Q(t)}{\mathcal{E} \cdot C} = e^{-t/RC}$$

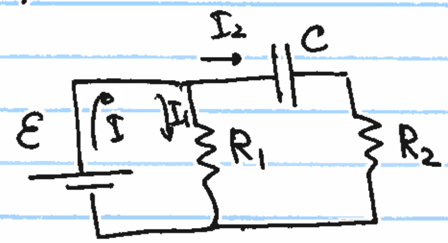
$$Q(t) = C \cdot \mathcal{E} - C \cdot \mathcal{E}\, e^{-t/RC} = C \cdot \mathcal{E}\left(1 - e^{-t/RC}\right)$$



$$I(t) = \frac{dQ}{dt} = \frac{\mathcal{E}}{R} e^{-t/RC}$$



A bit more complicated circuit



Kirchhof's rule

$I(t) = I_1(t) + I_2(t)$

$\mathcal{E} = I_1 R_1 \qquad I_1 = \mathcal{E}/R_1$ at any time

$\mathcal{E} - \frac{Q(t)}{C} - I_2(t)\cdot R_2 = 0$

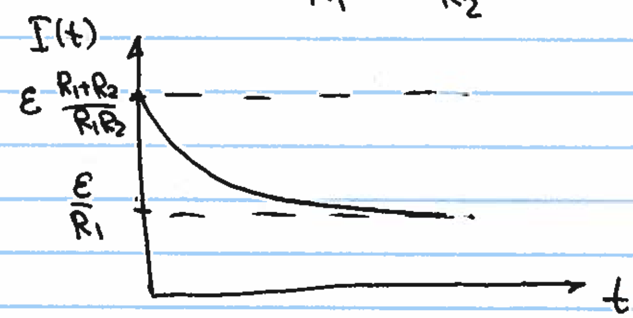
$I(t{=}0) = \mathcal{E}\,\frac{R_1+R_2}{R_1 R_2}$

$I(t{=}\infty) = \mathcal{E}/R_1$

This is the same eqn we just solved!

$$I_2(t) = \frac{\mathcal{E}}{R_2}\, e^{-t/R_2 C}$$

$$I(t) = \frac{\mathcal{E}}{R_1} + \frac{\mathcal{E}}{R_2}\, e^{-t/R_2 C}$$



From the same solution

$Q(t) = C\cdot\mathcal{E}(1 - e^{-t/R_2 C})$

the capacitor is charging