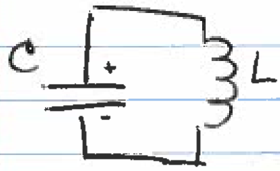
Written assignment #8 solutions

Q_1



$$V_C = \frac{Q}{C} \qquad V_L = -L\frac{dI}{dt} = L\frac{d^2Q}{dt^2}$$

$$V_C + V_L = 0$$

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

a) $Q(t) = Q_0 \cos\omega_0 t \qquad -L\omega_0^2 + \frac{1}{C} = 0$

$I(t) = -\omega_0 Q_0 \sin\omega_0 t \qquad \omega_0 = 1/\sqrt{LC}$

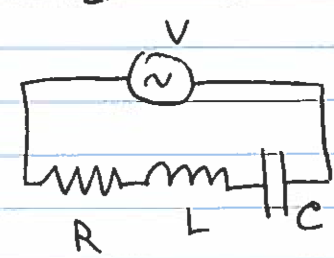
does not depend on the initial charge

b) $L\cdot C = 1/\omega_0^2 = \frac{1}{(2\pi)^2}\frac{1}{(10^7 Hz)^2} \approx 2.5\cdot 10^{-16}\ 1/Hz^2$

Typical capacitors – 10pF (10^{-11}F) to 1mF (10^{-3}F)
Typical inductances – 0.1μH (10^{-7}F) to 10mH (10^{-2}H)

(For example C = 50pF and L = 5μH will work)

Q2



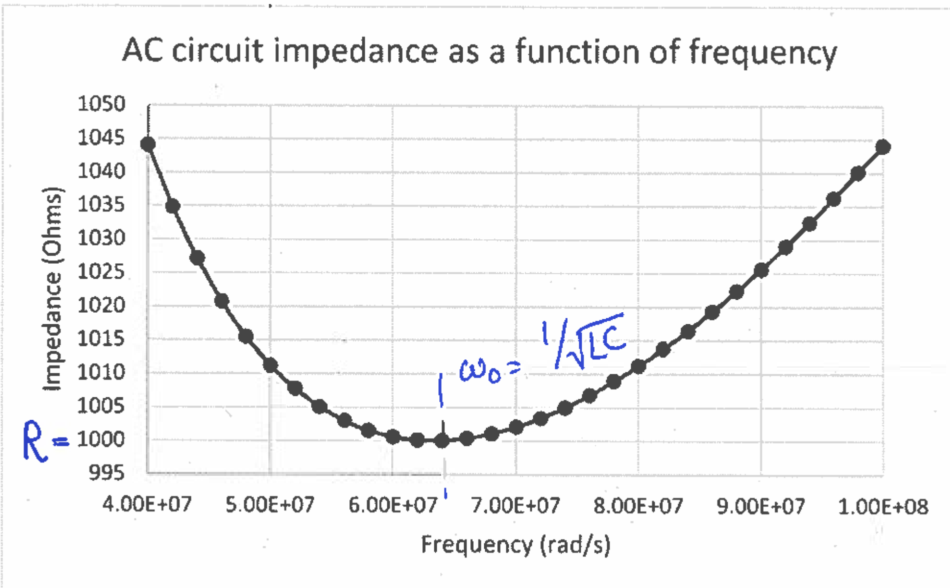
$V(t) = V_0 e^{i\omega t} \qquad I(t) = I_0 e^{i\omega t}$
(V_0, I_0 can be complex!)

a) $V_R = -IR$, $V_L = -L\frac{dI}{dt} = -iL\omega I_0 e^{i\omega t}$

$$V_C = Q/C = -\frac{1}{C}\int I_0 e^{i\omega t} dt = -\frac{1}{i\omega C} I_0 e^{i\omega t}$$

$$\cancel{V(t)=}\ V_0 e^{i\omega t} - I_0 R e^{i\omega t} - i\omega L I_0 e^{i\omega t} + \frac{1}{i\omega C} I e^{i\omega t} = 0$$

$$V(t) = \underbrace{\left(R + i\omega L + \frac{1}{i\omega C}\right)}_{Z(\omega)} I(t) \qquad |Z(\omega)| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$


AC circuit impedance as a function of frequency
Impedance (Ohms)
Frequency (rad/s)
1050
1045
1040
1035
1030
1025
1020
1015
1010
1005
1000
995
4.00E+07
5.00E+07
6.00E+07
7.00E+07
8.00E+07
9.00E+07
1.00E+08
R =
$\omega_0 = 1/\sqrt{LC}$

Q3

$$\frac{d^2x}{dt^2} + \frac{2}{\tau}\frac{dx}{dt} + \omega_0^2 x = -\frac{eE_0}{2m} e^{-i\omega t}$$

a) $x = x_0 e^{-i\omega t}$

$$-\omega^2 x_0 \cancel{e^{-i\omega t}} + \frac{2}{\tau}(-i\omega) x_0 \cancel{e^{-i\omega t}} + \omega_0^2 x_0 \cancel{e^{-i\omega t}} = -\frac{eE_0}{2m} \cancel{e^{-i\omega t}}$$

$$\left[(\omega_0^2 - \omega^2) - \frac{2i\omega}{\tau} \right] x_0 = -\frac{eE_0}{2m}$$

$$x(t) = -\frac{eE_0}{2m} e^{-i\omega t} \frac{1}{(\omega_0^2-\omega^2) - 2i\omega/\tau}$$

b) if $\omega_0 \approx \omega$ $\quad (\omega_0^2 - \omega^2) = (\omega_0-\omega)(\omega_0+\omega) \approx 2\omega(\omega_0-\omega)$

$$x(t) \approx -\frac{eE_0}{4m\omega} e^{-i\omega t} \frac{1}{(\omega_0-\omega) - i/\tau} = -\frac{ieE_0\tau e^{+i\omega t}}{1+i(\omega_0-\omega)\tau}$$

$$d(t) = -e\,x(t) = +\frac{1}{4m\omega}\frac{ie^2E_0\tau}{1+i(\omega_0-\omega)\tau} e^{-i\omega t} = \frac{ie^2E_0\tau e^{-i\omega t}}{4m\omega(1+i(\omega_0-\omega)\tau)}$$

Q4

$$\chi(\omega) = \frac{\varepsilon_0 e^2 N\tau}{4im\omega}\frac{1}{1+i(\omega_0-\omega)\tau} = \frac{\varepsilon_0 e^2 N\tau}{4m\omega}\frac{(+i)(1-i\tau(\omega_0-\omega))}{1+\tau^2(\omega_0-\omega)^2}$$

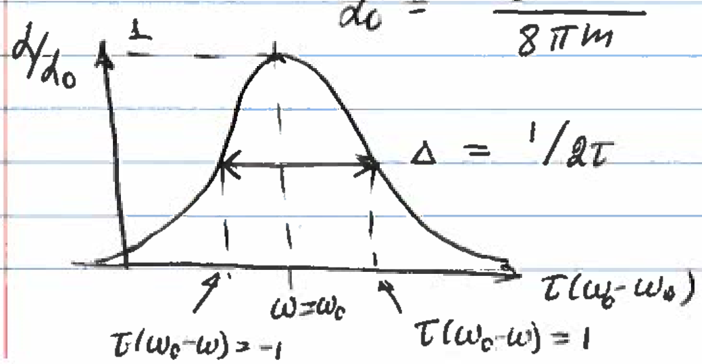
$$= \frac{\varepsilon_0 e^2 N\tau}{4m\omega}\frac{+i+\tau(\omega_0-\omega)}{1+\tau^2(\omega_0-\omega)^2}$$

a) Absorption $\quad \alpha(\omega) = +\frac{1}{\lambda}\chi(\omega) = +\frac{\omega}{2\pi c}\cdot\frac{\varepsilon_0 e^2 N\tau}{4m\omega}\frac{1}{1+\tau^2(\omega_0-\omega)^2}$

$$\alpha(\omega) = \frac{\varepsilon_0 e^2 N\tau}{8\pi m}\frac{1}{1+\tau^2(\omega_0-\omega)^2}$$

max absorption $\omega = \omega_0$ (on resonance)

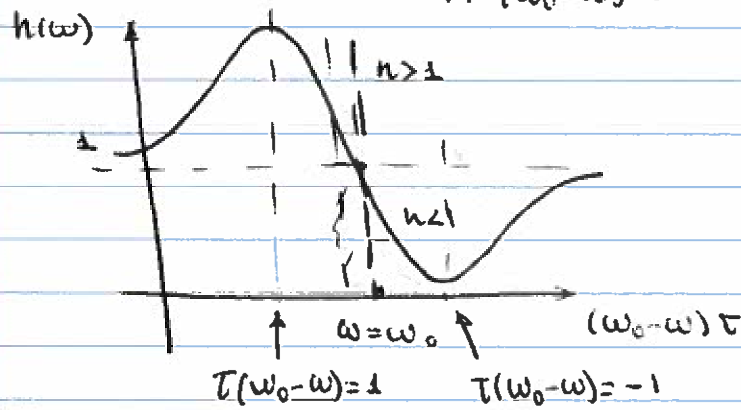
$$\alpha_0 = \frac{\varepsilon_0 e^2 N\tau}{8\pi m}$$



The longer is the liftime of an excited atomic state, the narrower is the absorption resonance

b) $n(\omega) = 1 + \mathrm{Re}(\chi(\omega)) = 1 + \frac{\varepsilon_0 e^2 N\tau}{4m\omega} \frac{\tau(\omega_0-\omega)}{1+\tau^2(\omega_0-\omega)^2} =$

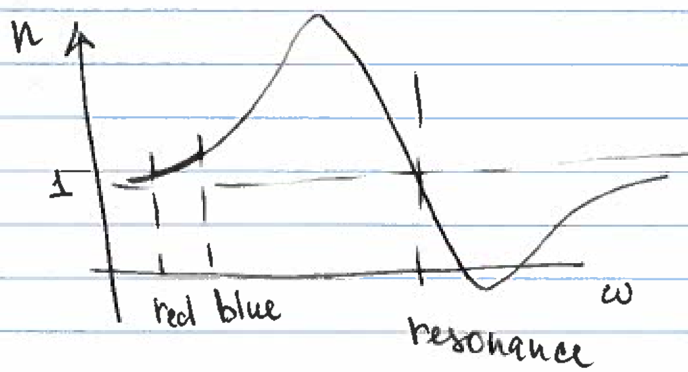
$$= 1 + \lambda\alpha_0 \frac{(\omega_0-\omega)\tau}{1+(\omega_0-\omega)^2\tau^2}$$



On resonance we have $\frac{dn}{d\omega} < 0$ (negative dispersion)

outside of absorption the dispersion is positive $dn/d\omega > 0$

c) $n_{red} = 1.5$, $n_{blue} = 1.54$



Since $1 < n_{red} < n_{blue}$ we expect to find resonances in glass in UV