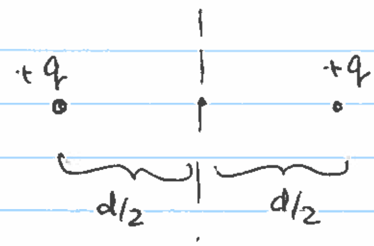
Written assignment #4 solutions

Q1 1D ion trap



a) If a positive charge is placed exactly b/w two tpapp changes, it will experience no net force

If the charge shifts by x in one of the directions

$$F_{Net} = \frac{kq\,q_{test}}{(d/2+x)^2} - \frac{kq\,q_{test}}{(d/2-x)^2}$$

b) if $x \ll d$, then $\dfrac{1}{(d/2 \pm x)^2} = \dfrac{4}{d^2}\dfrac{1}{(1 \pm 2x/d)^2} \approx \dfrac{4}{d^2}\left(1 \mp \dfrac{4x}{d}\right)$

Thus, for small displacement

$$F_{Net} \approx kq\,q_{test}\left[\left(\frac{4}{d^2} - \frac{16x}{d^3}\right) - \left(\frac{4}{d^2} + \frac{16x}{d^3}\right)\right] = \cancel{-\frac{16 kq\,q_{test}}{d^3}x}$$

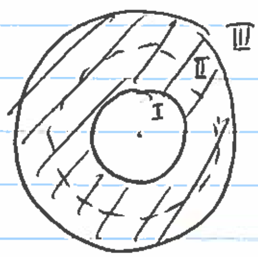
$$= -\frac{32 kq\,q_{test}}{d^3} \cdot x$$

same form as in Hooke's law for the restorative force $F = - k_{Hooke} \cdot x$

where $k_{Hooke} = \dfrac{32 kq \cdot q_{test}}{d^3} = 7.36e^{-17}$ N/m

c) Oscillation frequency $f = \dfrac{1}{2\pi}\sqrt{k/m_e} = 1.43$ MHz

Q2



"Donut" volume

$\frac{4\pi}{3}R_2^3 - \frac{4\pi}{3}R_1^3$

a) Charge density $\rho = \frac{Q}{\frac{4\pi}{3}(R_2^3 - R_1^3)}$

I, inside the ~~whole~~

$E = 0$, no enclosed charges

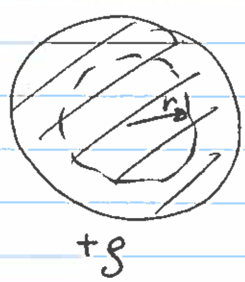
II $\quad \Phi = 4\pi r^2 \cdot E(r) = \frac{q_{enc}}{\varepsilon_0}$

$$q_{enc} = \rho\left(\frac{4\pi r^3}{3} - \frac{4\pi R_1^3}{3}\right)$$

$$\vec{E} = \frac{\hat{r}}{4\pi\varepsilon_0 r^2}\,\frac{\cancel{\frac{4\pi}{3}}(r^3 - R_1^3)}{\cancel{\frac{4\pi}{3}}(R_2^3 - R_1^3)} = \frac{\hat{r}}{4\pi\varepsilon_0 r^2}\left(\frac{r^3 - R_1^3}{R_2^3 - R_1^3}\right)$$

III $\quad \Phi = 4\pi r^2 \cdot E(r) = \frac{Q}{\varepsilon_0} \qquad \vec{E}(r) = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{r}$

b)



$$4\pi r^2 E_+ = \frac{4\pi}{3}\rho r^3 \cdot \frac{1}{\varepsilon_0}$$

$$E_+ = \frac{1}{4\pi\varepsilon_0}\, r\left(\frac{4\pi}{3}\rho\right)$$

$$4\pi r E_- = -\frac{4\pi}{3}\rho R_1^3 \cdot \frac{1}{\varepsilon_0}$$

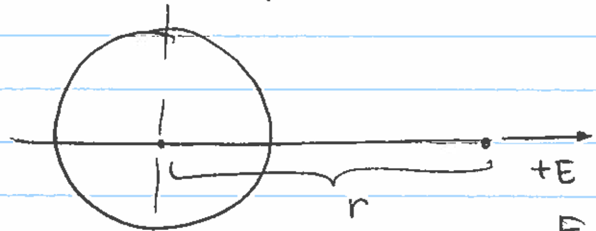
$$E_- = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{r^2}\left(-\frac{4\pi}{3}\rho\right)R_1^3$$

$$E(r) = E_+ + E_- = \frac{1}{4\pi\varepsilon_0}\left(\frac{4\pi\rho}{3}\right)\left(r - \frac{R_1^3}{r^2}\right) =$$

$$= \frac{1}{4\pi\varepsilon_0 r^2}\left(\cancel{\frac{4\pi}{3}} \cdot \frac{Q}{\cancel{\frac{4\pi}{3}}(R_2^3 - R_1^3)}\right)(r^3 - R_1^3) =$$

$$= \frac{Q}{4\pi\varepsilon_0 r^2}\left(\frac{r^3 - R_1^3}{R_2^3 - R_1^3}\right)$$

Q3 Again, we present the shell as two solid spheres

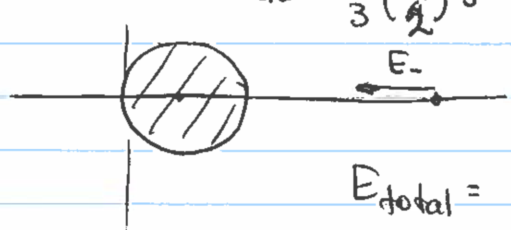


$$E_+ = \frac{kQ_+}{r^2} = \frac{4\pi}{3} k\rho \frac{R^3}{r^2}$$

total charge

$$Q_+ = \frac{4\pi}{3} R^3 \cdot \rho$$

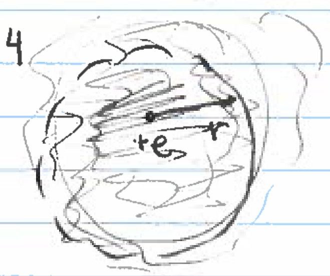
$$Q_- = \frac{4\pi}{3}\left(\frac{R}{2}\right)^3 \cdot \rho$$



$$E_- = -\frac{kQ_-}{(r-R/2)^2} = -\frac{4\pi}{3} k\rho \frac{(R/2)^3}{(r-R/2)^2}$$

$$E_{total} = \frac{4\pi}{3} k\rho R^3 \left[\frac{1}{r^2} - \frac{1}{8(r-R/2)^2}\right]$$

Q4



$$E(r) = \underbrace{\frac{ke}{r^2}}_{\text{proton}} + E_{electron}$$

Gauss's law

$$4\pi r^2 E_{electron} = \frac{q_{enc}}{\varepsilon_0} = \frac{(-e)}{\varepsilon_0}\int_0^r \frac{1}{\pi a_0^3} e^{-2r'/a_0} 4\pi r'^2 dr'$$

$$= \frac{4}{\varepsilon_0}(-e)\int_0^r \frac{1}{a_0^3} e^{-2r'/a_0} r'^2 dr' = \frac{4}{\varepsilon_0}(-e)\int_0^{r/a_0} e^{-2(r'/a_0)} (r'/a_0)^2 d(r'/a_0) =$$

$$= \frac{4}{\varepsilon_0}(-e)\int_0^{r/a_0} e^{-2x} x^2 dx = \frac{4(-e)}{\varepsilon_0}\left[-\frac{1}{4} e^{-2x}(2x^2+2x+1)\right]\Bigg|_0^{r/a_0} =$$

$$= \frac{(-e)}{\varepsilon_0}\left[1 - e^{-2r/a_0}(2r^2/a_0^2 + 2r/a_0 + 1)\right]$$

$$E_{electron} = \frac{(-e)}{4\pi\varepsilon_0}\left[\frac{1}{r^2} - \frac{1}{r^2} e^{-2r/a_0}(2r^2/a_0^2 + 2r/a_0 + 1)\right]$$

$$E_{total} = E_{proton} + E_{electron} = \frac{e}{4\pi\varepsilon_0 r^2} e^{-2r/a_0}(2r^2/a_0^2 + 2r/a + 1)$$