Coherent states

\[ E_x = i \sqrt{\frac{\hbar \omega}{2\mathcal{E}_0}} \left( \hat{a}e^{ikz - i\omega t} - \hat{a}^\dagger e^{-ikz + i\omega t} \right) \]

The number states we've discussed are highly non-classical, i.e., they do not have classical analogues.

Measured mean amplitude of e.m. field

\[ \langle E_x \rangle = \langle n|E_x|n \rangle = i\sqrt{\frac{\hbar \omega}{2\mathcal{E}_0}} \left( \langle n|\hat{a}e^{ikz - i\omega t} \rangle - \langle n|\hat{a}^\dagger e^{-ikz + i\omega t} \rangle \right) = 0 \]

no well-defined amplitude

Fluctuations of electro-magnetic field

\[ \Delta E_x = \sqrt{\langle E_x^2 \rangle - \langle E_x \rangle^2} = \sqrt{\frac{\hbar \omega}{2\mathcal{E}_0}} \sqrt{\langle n|\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger |n \rangle} = \sqrt{\frac{\hbar \omega}{2\mathcal{E}_0}} \sqrt{2n+1} \]

Note that even for \( \left| 0 \right> \) vacuum state

\[ \Delta E_x = \sqrt{\frac{\hbar \omega}{2\mathcal{E}_0}} > 0 \]

Vacuum fluctuations (now vacuum is not nothingness, it is alive and wiggles)

So what state would be the closest analogue of the classical e.m. wave?
Coherent states are the eigenstates of the annihilation operator.

\[ | \alpha \rangle = \alpha | 0 \rangle \]

If \[ | 0 \rangle = \sum_{n=0}^{\infty} c_n | n \rangle \Rightarrow \alpha | 0 \rangle = \sum_{n=0}^{\infty} c_n \frac{\alpha^n}{\sqrt{n!}} | n \rangle \]

\[ c_{n+1} \frac{\alpha}{\sqrt{(n+1)!}} = \alpha c_n \frac{\alpha^n}{\sqrt{n!}} \]

\[ \Rightarrow c_n = \frac{\alpha^n}{\sqrt{n!}} c_0 \]

\[ | 0 \rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} | n \rangle \]

\[ \langle \alpha | 0 \rangle = 1 = | 0 \rangle \]

\[ \langle \alpha | 0 \rangle = \frac{1}{\sqrt{\alpha^2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} | n \rangle \]

\[ 1 \alpha^2 e^{-\frac{\alpha^2}{2}} = 1 \Rightarrow 1 \alpha^2 = e^{-\frac{\alpha^2}{2}} = c_0 \]

\[ | 0 \rangle = e^{-\frac{\alpha^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} | n \rangle \]

Average value of the electric field:

\[ \langle d | E_x | d \rangle = i \sqrt{\frac{\hbar \omega}{2\epsilon_0 V}} \left( \frac{\langle d | \hat{a} | d \rangle e^{ikz-\omega t} - \langle d | \hat{a}^+ | d \rangle e^{-ikz+\omega t}}{\alpha} \right) \]

\[ = i \sqrt{\frac{\hbar \omega}{2\epsilon_0 V}} \left( d e^{ikz-\omega t} - d^* e^{-ikz+\omega t} \right) = \]

\[ \alpha = | d | e^{i\varphi} \quad d^* = | d | e^{-i\varphi} \]

\[ = 2|d| \sqrt{\frac{\hbar \omega}{2\epsilon_0 V}} \sin (kz - \omega t + \varphi) \]

Total energy \[ \frac{1}{2} \int_0^\infty E_1^2 dV = \frac{1}{2} |d|^2 \hbar \omega \]

"Average" number of photons

\[ \langle d | \hat{a}^+ \hat{a} | d \rangle = |d|^2 \]
Photon distribution in coherent states with different mean value of photon 1d1.
Electric Field Fluctuations

\[ \langle d l E^2 l d \rangle = -\left( \frac{h \omega}{2 \varepsilon_0 V} \right) \left( d l \hat{a}^\dagger e^{2ikz-\text{int}} + \hat{a} e^{-2(kz-\text{int})} \right) \]

\[ = -\left( \frac{h \omega}{2 \varepsilon_0 V} \right) \left( d l^2 e^{2ikz-\text{int}} + d l^2 e^{-2(kz-\text{int})} - 1 - 2d l^2 \right) \]

\[ = -\left( \frac{h \omega}{2 \varepsilon_0 V} \right) \left( \frac{2d l^2 \cos(2kz-\omega t + \phi) - 2d l^2 - 1}{-4d l^2 \sin^2(kz-\omega t + \phi)} \right) \]

\[ \langle d l E^2 l d \rangle = \left( \frac{h \omega}{2 \varepsilon_0 V} \right) \left( 1 + 4d l^2 \sin^2(kz-\omega t + \phi) \right) \]

\[ \Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2} = \sqrt{\frac{h \omega}{2 \varepsilon_0 V}} \]

Same fluctuation as in vacuum state

Coherent state is a minimum uncertainty state.

Coherent state is a displaced vacuum state

\[ |d> = D(d) |0> \]

Displacement operator

\[ D(d) = e^{d \hat{a} - d^* \hat{a}^\dagger} \]

\[ \hat{D} \text{ is a unitary operator} \]

\[ D^d(d) B^+ (d) = B^+ (d) D(d) = 1 \]

since

\[ D^d(d) = (e^{d \hat{a} - d^* \hat{a}^\dagger})^+ = e^{-d^* \hat{a}^\dagger - d \hat{a}} = D(-d) \]
**Coherent state =**

"fuzzy" electromagnetic wave
Increasing strength of e-m field

Noise current $i_\Omega$ in units of $\Delta E_{\text{vac}}$

Since the uncertainty effect becomes less noticeable as the amplitude grows, it stays the same as the strength of the e-m field increases.
Vacuum Fluctuations

For a single mode, $E_m = \hbar \omega (n + \frac{1}{2})$

For a vacuum state, $n = 0$

Zero point energy,

$$E_{\text{vac}} = \sum_{\text{mode}} \frac{1}{2} \hbar \omega = 0$$

(Too) Simple solution $\rightarrow$ renormalization

(just shift the level from which
we count energy)

The effect of fluctuations is
directly observable

a) Spontaneous emission: electron in
excited states interact with vacuum
fluctuations and, as a result, change
their energy level, emitting
thermal radiation

b) Since there is always uncertainty
in measurable e-m field
amplitude, all optical measurements
are fundamentally limited in
precision

c) Lamb shift

Experimentally, $2S_{1/2}$ and $2P_{1/2}$ states
in H atom are split by $\sim 1 \text{GHz}$

In a semiclassical approximation,
they must be degenerate.
Vacuum fluctuations make an electron do randomly fluctuate from its equilibrium position, changing its energy in the Coulomb potential

\[ \Delta E = \frac{1}{6} \langle \Delta r^2 \rangle \cdot 4\pi e^4 \frac{1}{\hbar^2} r_n \cdot (r=0) \cdot \frac{1}{(r=0)^2} = 0 \text{ for all states except } \]
\[ \text{for } l=0 \text{ (S state).} \]

d) Casimir Force

\[ E_{\text{ZPE}} = \sum_{\text{modes}} \frac{1}{2} k \omega_i \]

Two perfectly conducting parallel plates

if \( d \approx \lambda \), only the wave vector \( k_z = \frac{2\pi n}{d} \) are possible

\[ \omega_n = c k_n \implies c = \frac{\hbar k_n}{\lambda} \]

\[ E_{\text{ZPE}}^{(\text{in})} = \int dk_x dk_y dk_z \frac{1}{2} \hbar c \sqrt{k_x^2 + k_y^2 + (2\pi n/d)^2} \]

outside → no restrictions

\[ E_{\text{ZPE}}^{(\text{out})} = \int dk_x dk_y dk_z \left( \frac{1}{2} \hbar c \sqrt{k_x^2 + k_y^2 + k_z^2} \right) \]

\[ U = E_{\text{ZPE}}^{(\text{in})} - E_{\text{ZPE}}^{(\text{out})} = \text{alter tedious calculations} \]

\[ U = \frac{\pi^2 \hbar c}{720 d^3} L^2 \]

Casimir force (per unit area)

\[ F = \frac{1}{L^2} \frac{dU}{dd} = -\frac{\pi^2 \hbar c}{240 d^4} \]