Light - atom interaction

1. Hamiltonian

Pure atomic system: \( \hat{H}_0 = \frac{1}{2m} \hat{P}^2 + V(r) \) provides atomic structure

In the presence of the external fields

\[
\hat{H} = \frac{1}{2m} [\hat{\mathbf{p}} + e \hat{A}]^2 - e \varphi + V_c(r)
\]

\[\hat{\mathbf{p}} \rightarrow \mathbf{p}, \quad \text{vector potential} \quad \hat{\varphi} \rightarrow \varphi, \quad \text{electrostatic potential}\]

Coulomb (or radiation) gauge: \( \nabla \cdot \hat{A} = 0, \quad \varphi = 0 \)

In this case

\[
\nabla^2 \hat{A} - \frac{1}{c^2} \frac{\partial^2 \hat{A}}{\partial t^2} = 0
\]

\[
\hat{H} = \frac{1}{2m} [\hat{\mathbf{p}} + e \hat{A}]^2 + V(r) \approx \frac{1}{2m} \hat{\mathbf{p}}^2 + V(r) + \frac{e}{m} \hat{\mathbf{A}} \cdot \hat{\mathbf{p}} + \frac{e^2}{2m} \hat{\mathbf{A}}^2
\]

\( \hat{\mathbf{p}} \rightarrow \mathbf{p}, \quad \text{atomic structure} \quad \hat{V} \rightarrow V, \quad \text{perturbation} \)

Dipole approximation: \( \lambda \gg a \) (extend of the electron's wave function)

\[
\hat{H} = \frac{1}{2m} \hat{\mathbf{p}}^2 + V_c(r) + e \hat{\mathbf{F}} \cdot \hat{\mathbf{E}}
\]

\( \hat{H}_0 \quad \hat{H}_I \) interaction hamiltonian
2. Interaction of an atom with a classical field (perturbative approach)

We are going to treat the interaction with e-m field as perturbation. Thus, its effect will enable transitions b/w various states

\[ |\Psi(t)\rangle = \sum_{k} c_k(t) e^{-iE_k t/\hbar} |k\rangle \]

\[ \text{time-dependent perturbation theory} \]

\[ \frac{\partial |\Psi\rangle}{\partial t} = (\hat{H}_0 + \hat{H}_I) |\Psi(t)\rangle \]

\[ c_k(t) = -\frac{i}{\hbar} \sum_{k} c_{k'}(t) \langle k'|\hat{H}_I|k\rangle e^{i\omega_{kk'} t} \]

\[ \omega_{kk'} = \frac{E_k - E_{k'}}{\hbar} \text{ transition freq.} \]

The situation we are going to consider is that at some point of time our system is initialized \( \rightarrow \) we know the exact initial state \( |i\rangle \Rightarrow c_i = 1, c_{i+1} = 0 \)

If no interaction, an atom would stay in this state forever (since it is the eigenstate of \( \hat{H}_0 \))

First order perturbation theory

\[ C(t) = C_{(0)}(t) + C_{(1)}(t) + \ldots \]

\[ \text{no interaction} \sim \hat{H}_I; \text{ (we neglect)} \]

\[ C_{(0)}(t) = 1 \quad C_{(1)}(t) = 0 \]
\[ \mathbf{c}_i(t) = -\frac{i}{\hbar} \langle i|\hat{H}_\perp|i \rangle \]
\[ \mathbf{c}_f(t) = -\frac{i}{\hbar} \langle f|\hat{H}_\perp|i \rangle e^{i\omega t} \]

\[ \hat{H}_\perp(t) = e^2\overline{E_0} \cos \omega t = -\overline{\dot{E}_0} \cos \omega t \]

Transition dipole moment \( \langle f|\overline{\dot{E}_0}|i \rangle = \delta_{fi} \overline{E_0} \)
determines the selection rules
if \( \delta_{fi} = 0 \) transition is forbidden

Since \( \overline{\dot{E}_0} \) is parity-odd, the state \( f \) and \( i \)
must have opposite parity to make possible
electro-dipole transition. It also means
\( \langle i|\hat{H}_\perp|i \rangle = 0 \) for any \( |i \rangle \), \( \mathbf{c}_i^{(0)} = 0, \mathbf{c}_i^{(1)} = 0 \)
Thus, in the first approximation the initial state
is unperturbed.

\[ \mathbf{c}_f^{(1)} = -\frac{i}{\hbar} \int_0^\infty (\delta_{fi} \cdot \overline{E_0}) \cos \omega t \ e^{i\omega t} dt = \]
\[ = -\frac{i}{2\hbar} (\delta_{fi} \cdot \overline{E_0}) \left( e^{i(w+\omega_f)t} + e^{-i(w+\omega_f)t} \right) dt = \]
\[ = - (\delta_{fi} \overline{E_0}) \frac{1}{2\hbar} \left( \frac{e^{i(w+\omega_f)t} - e^{-i(w+\omega_f)t}}{w+\omega_f} \right) \]

Rotating wave approximation \( |w+\omega_f| \ll \omega, \omega_f \)
\( \rightarrow \) we can neglect the first term

\[ \mathbf{c}_f^{(1)} = (\delta_{fi} \overline{E_0}) \frac{e^{-i(w+\omega_f)t} - 1}{2\hbar (w+\omega_f)} \]

\[ P_{fi} = |\mathbf{c}_f^{(1)}|^2 = \frac{18 \delta_{fi} \overline{E_0}^2}{\hbar^2} \left( \sin \frac{(w+\omega_f)t}{2} \right)^2 = \frac{18 \delta_{fi} \overline{E_0}^2 \sin^2 \frac{\theta}{2}}{\hbar^2} \Delta^2 \]

\( \Delta = w-\omega_f \) - detuning of
the laser from the atomic resonance
For $\Delta \neq 0$ (non-resonant conditions)

$$P_{fi} \leq \frac{1}{8\pi^2} \frac{\lambda}{h^2} \frac{1}{\Delta^2}$$

For $\Delta \to 0$ \hspace{1cm} $\sin \frac{\Delta t}{2} \to \frac{\Delta t}{2}$

$$P_{fi} \bigg|_{\Delta=0} = \frac{1}{8\pi^2} \frac{\lambda}{h^2} t^2 \quad \text{not physical for large } t, \text{ since } P \leq 1$$

It is more accurate to define the transition probability rate $W_{fi} = \frac{dP_{fi}}{dt}$

We will use \lim_{t \to \infty} \frac{\sin^2(\frac{\Delta t}{2})}{\Delta^2} = \frac{\pi}{2} \delta(\Delta)

Thus we can present $P_{fi}' = \frac{1}{8\pi^2} \frac{\lambda}{h^2} \frac{\pi}{2} \delta(\Delta)$

and $W_{fi} = \frac{1}{8\pi^2} \frac{\lambda}{h^2} \frac{\pi}{2} \delta(\Delta)$

If there are many possible final states, then the total probability to leave the initial state $|i\rangle$ is

$$W_{i \to \text{ final}} = \int \frac{1}{8\pi^2} \frac{\lambda}{h^2} \frac{\pi}{2} \delta(w - W_{fi})$$

Fermi's golden rule