Quantum Rabi Flopping

\[ |\psi(t)\rangle = C_a(t) |a, n+1\rangle + C_b(t) |b, n\rangle \]

Let \[ C_b(t=0) = 1, \quad C_a(t=0) = 1 \]

\[ i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi \]

gives common phase factors

\[ i\hbar (\dot{C_b}) = \hbar \omega (n+\frac{1}{2}) (C_b) + \frac{1}{2\hbar} (g^2 n^2 + 2g^2 n + g^2) \frac{C_b}{C_a} \]

\[ i\hbar (\dot{C_a}) = \frac{\hbar}{2} g^2 n \frac{C_a}{C_b} \]

For simplicity

\[ C_b + g^2 (n+1) C_a = 0 \]

\[ C_b = \cos (g\sqrt{n+1} t) \quad P_b = \cos^2 (g\sqrt{n+1} t) \]

\[ C_a = -i \sin (g\sqrt{n+1} t) \quad P_a = \sin^2 (g\sqrt{n+1} t) \]

\[ |\psi(t)\rangle = \cos (g\sqrt{n+1} t) |b, n\rangle - i \sin (g\sqrt{n+1} t) |a, n+1\rangle \]

Quantum Rabi Flopping

Notice that even if \( n = 0 \) (vacuum), there still will be sloppings with frequency \( 2g \) (vacuum Rabi Flopping)
So, a Fock state with a fixed number of photons $\left| n \right>$ behaves very similarly to a classical Rabi flipping.

What about a coherent state?

Initially

$$\left| \psi_{\text{atom}} \right>_0 = c_a \left| a \right> + c_b \left| a \right>$$

$$\left| \psi_{\text{light}} \right>_0 = \sum_{n=0}^{\infty} c_n \left| n \right> \quad c_n = e^{-\frac{1}{2} \left| n \right|^2}$$

for a coherent state

$$\left| \psi(t=0) \right> = \left| \psi_{\text{atom}} \right>_0 \left| \psi_{\text{light}} \right>_0$$

As we discussed before, light-atom interaction couples states $\left| a, n+1 \right>$ and $\left| b, n \right>$ for all $n$ present.

$$\left| \psi(t) \right> = \sum_{n=0}^{\infty} \left[ c_b c_n \cos \left( gt - \sqrt{n+1} \right) - i c_a c_{n+1} \sin \left( gt - \sqrt{n+1} \right) \right] \left| a \right> + \sum_{n=0}^{\infty} \left[ -i c_b c_{n-1} \sin \left( gt - \sqrt{n} \right) + c_a c_n \cos \left( gt - \sqrt{n} \right) \right] \left| a \right>$$

For $c_b = 1$ (we start with an atom in the excited state)

$$\left| \psi(t) \right> = \sum_{n=0}^{\infty} \left[ e_n \cos \left( gt - \sqrt{n+1} \right) \left| b \right> - i e_{n-1} \sin \left( gt - \sqrt{n} \right) \left| a \right> \right] \left| n \right>$$

$$\left| \psi_a(t) \right> = \sum_{n=0}^{\infty} e_n \sin \left( gt - \sqrt{n+1} \right) \left| n+1 \right> \quad \text{ground}$$

$$\left| \psi_b(t) \right> = \sum_{n=0}^{\infty} e_n \cos \left( gt - \sqrt{n+1} \right) \left| n \right> \quad \text{excited}$$
Average atomic inversion

\[ \langle \psi(t) | \hat{\sigma}_z | \psi(t) \rangle = \langle \psi(t) | \hat{a} \rangle \langle \hat{a}^\dagger - 1 | a \rangle \langle a | | \psi(t) \rangle \]

\[ = \langle \psi_b | \psi_b \rangle - \langle \psi_a | \psi_a \rangle = \sum_{n=0}^{\infty} |c_{n}\rangle \langle c_{n}| \cos(2ngt + \tau n+1) - \sin^2(gt + \tau n+1)\]

\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \cos(2gt + \tau n+1) \]

\[ = e^{-\frac{1}{2g}\int_{1/d1}^{n} \cos(2gt + \tau n+1) \]

The output is a combination of many sine waves with somewhat different periods → no clear Rabi fluctuations

Main contributing components lie between frequencies \(2g \sqrt{n - \delta n}\) and \(2g \sqrt{n + \delta n}\)

Corresponding phase spread

\[ 2gt_c (\sqrt{n + \delta n} - \sqrt{n - \delta n}) \approx 2gt_c \sqrt{n} \left( (1 + \frac{\delta n}{2n}) - (1 - \frac{\delta n}{2n}) \right) \]

\[ \approx 2gt_c \frac{\delta n}{2n} \ll 1 \implies gt_c \ll 1 \]

\[ t_c \approx \frac{1}{g} \text{ depends only on coupling strength} \]
However we can also expect to see a revival of Rabi oscillations if

\[ (g\sqrt{n+1} - g\sqrt{n}) \frac{1}{\tau_R} = 2\pi k \quad (k=0, 1, 2, \ldots) \]

\[ g\sqrt{n} \left( 1 + \frac{1}{2n} - 1 \right) \frac{1}{\tau_R} = 2\pi \quad (k=1 \text{ for the first occurrence}) \]

\[ g \frac{1}{2\sqrt{n}} \frac{1}{\tau_R} = 2\pi \]

\[ \tau_R = \frac{4\pi \sqrt{n}}{g} \]

The revival is never "complete" since the frequencies \( g\sqrt{n} \) are not truly equidistant.

Why a coherent state is less "classical" than a number state?

Clear Rabi shogging requires knowledge of precise intensity. Coherent state as a minimum uncertainty state, has certain spread in its intensity distribution, that leads to the Rabi shogging diffusion.