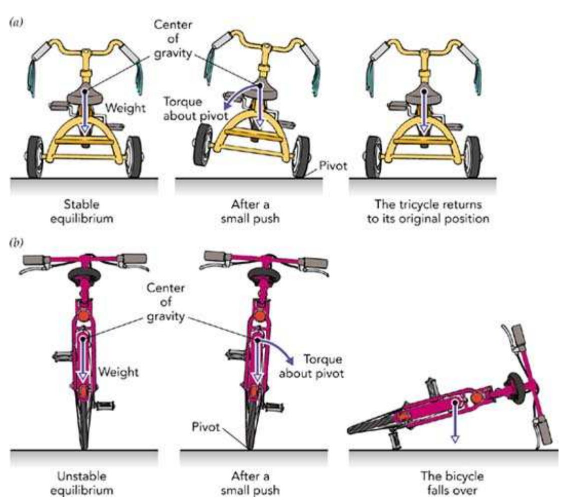
Bicycles and Stability

A bicycle is an amazing machine that demonstrates just about every principle of physics we have investigated so far. You do work on the pedal with your leg. This causes a torque on the crank sprocket. This turns a chain which causes torque on the back axis. By selecting different gears on the back axial you can apply efficient torque at different speeds. The wheels turn and the angular momentum of the turning tires provide (some) stability. The handlebar lets you turn the fork to redirect the front wheel to go in another direction. Squeeze the brake handle and the friction on the rim of the tire slows you down.



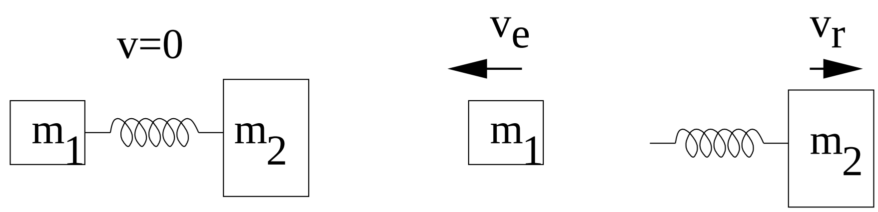
If takes some practice to ride a bike and not fall over. What do we mean by stability. Consider the tricycle. The center of gravity is inside the two rear wheels. If the tricycle starts to turn over, the center of mass of the tricycle and rider is inside the wheel base. The mass of the tricycle produces a torque to bring the tricycle back to its level stable position. A tilt produces an **increase** in the total gravitational potential energy. Now think about a bicycle. The smallest tilt and the torque produces by the weight of the bike and the rider produces a torque which rotates until it hits the ground. The small tilt produces a **decrease** in the gravitational energy

- **Stable Equilibrium:** An object is in a stable equilibrium when any small shift increases its total potential energy.
- **Unstable Equilibrium:** An object's equilibrium is unstable when a small shift can decrease its total potential energy.

Normally, we are interested in riding (moving) on a bicycle not just balancing on a stationary bike. The angular momentum ('gyroscope' effect) of the wheels give some stability but not much. What makes a bicycle stable is the placement of the front wheel. When the bicycle leans, the upward force of the pavement is no longer directly through the wheel's center of mass. This torque produces a '**precession**' of the wheel which attempts to return the wheel to the straight ahead direction. This is why riding a by cycle with hands in reasonably easy.

Rockets

A rocket or missile is propelled by ejecting material out the back. The reaction force of the exhaust provides the force to move the rocket forward. Conservation of momentum allow one to calculate the speed of the rocket at any point in its flight. Consider the very simple case of two masses connected by a compressed spring initially at rest. m_1 is the exhaust gas that will be expelled by the energy stored in the spring. m_2 is the rocket. If the spring is released, m_2 is pushed forward (right) and m_1 is pushed backwards (left).



Since momentum is conserved and initially zero, we have

$$\text{Momentum}_{initial} = \text{Momentum}_{final}$$

$$0 = \text{-m}_1 \text{ v}_e + \text{m}_2 \text{ v}_r$$

$$\text{v}_r = \text{v}_e \frac{m_1}{m_2}$$

You can see that the speed of the 'exhaust' determines the speed of the rocket. The calculation of the final 'ultimate' speed of a rocket requires more math than we want to use. The result is

$$\text{v}_r = \text{v}_e \ln\left(\frac{m_{rocket}+m_{fuel}}{m_{rocket}}\right)$$

where v_r is the final speed of the rocket, v_e is the speed of the exhaust, m_{rocket} is the mass of the rocket (without fuel) and m_{fuel} is the mass of the fuel exhausted. The 'ln' is the 'natural' logarithm (base e). To get the maximum speed, you need to have the exhaust exit the engine at the highest possible speed. Exhaust speeds for modern rocket fuels are several km/s. For liquid oxygen and hydrogen like the space shuttle, this speed is about 4.5 km/s. Ion propulsion has been explored. With an ion 'rocket' motor, the exhaust speed can be 30 km/s or more.

A common way to get more speed for the final satellite is the throw away the parts (stages) of the rocket during the flight. After you use that part of the rocket motor e.g. pumps, fuel tanks etc, you drop it off and start the next stage. This way, you only get the part of the rocket (the payload satellite) moving at the speed you want.

Satellites and Newton's Law of Universal Gravity

It is not true that Newton was hit on the head by an apple falling from a tree that inspired him to come up with the law of gravity! It is unlikely he was even under an apple tree when he did this incredible piece of science. For all of human existence, people had watched the planets and moon move in the night sky. In one simple equation Newton figured out how it all worked.

Everything in the universe pulls (but does not push) on everything else in the universe. Newton's law of universal gravitation is:

$$F \sim \frac{m_1 m_2}{d^2}$$

The gravitational force between any two objects is proportional to the product of their masses and inversely proportional to the distance between the two object's centers **squared**. If the mass is increased, the force increases proportionally. The force is reduced to $\frac{1}{4}$, if the distance between the objects doubles. Likewise if the distance is tripled, the force is $\frac{1}{9}$.

The Universal Gravitational Constant, G

So far we have just said the force is proportional to the masses and inversely proportional to the square of the distance between the masses' centers. To put numbers on the force in newtons, the masses in kilograms and the distance in meters, we have to figure out the constant of proportionality between the two sides of the equation. That constant is called the **Universal Gravitational Constant, G** and is:

$$G = 6.67 \text{ x } 10^{-11} \text{ N m}^2/\text{kg}^2$$

When we add this to the universal law of gravitation, we have:

$$F = G \, \frac{m_1 m_2}{d^2}$$

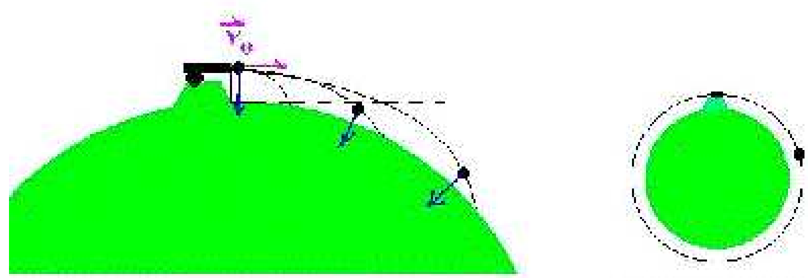
Notice how the units of G 'conspire' to make the units work out. The m^2 in the constant's numerator cancel the m^2 of the distance in the denominator. The kg^2 in the denominator of G cancel the two kg from the masses and we are left with just newtons which is the unit we want for your force.

G makes gravity a really weak force. You only notice gravity when you are near something very big like the earth. The force on a person from a 2 million kg ship is $\approx .006$ N. You would never notice the increased weight from even such a massive object.

Satellite Motion

When you throw a ball horizontally, it will eventually hit the ground. If you throw harder, it will travel further. The earth is a sphere so if you could throw really fast, you might get the ball over the horizon. For about every 8 km, the curvature of the earth drops 5 m. It takes a ball at the earth's surface 1 second to drop 5 meters so if you throw a ball that fast (8 km/s) the ball will follow the curvature of the earth. (Of course, if we did this in the atmosphere, the ball would burn up from air friction like a meteorite).

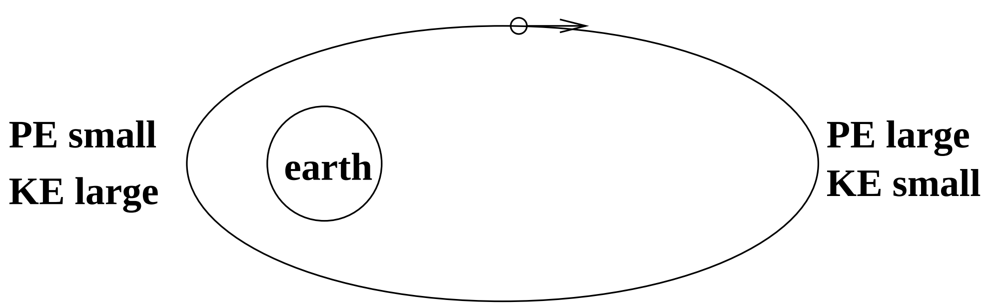
We say 'horizontal' what we really mean is tangent to the earth's surface. Once you start moving around the circular shape of the earth, horizontal means parallel to the earth's surface and vertical means away from the center of the earth.



If the tangential speed is 8 km/s and we get above the atmosphere, the object will move in a circular orbit around the earth. If we give it a bit more tangential speed, the object will move in an ellipse (an oval) or a higher circular orbit. The height of the orbit determines how long it takes for one period (one complete circle). If you put a satellite around the equator with an altitude of 35,900 km (22,300 miles), the period is 24 hours. Since the earth turns once every 24 hours, the satellite appears to be stationary above that point on the earth. Most communications satellites are placed in these 'geosynchronous' orbits. That is why your TV satellite dish always points roughly south.

We can understand elliptical orbits better with the help of energy conservation. In a circular orbit, the distance from the satellite to the center of the earth is constant so the potential energy is a constant. The satellite has a fixed speed (without air friction) so the kinetic energy is also constant.

For an elliptical orbit, energy conservation also can help us to understand the motion. The distance to the center of the earth changes so the PE changes. When the satellite is nearer to the earth's center (perigee), its speed and KE is larger since the total (mechanical) energy is a constant. At the maximum altitude from the earth (apogee), the satellite has it's smallest speed. When it is nearest the earth, it is moving with the fastest speed.



Escape Speed

There is an old say 'what goes up, must come down'. If you send an object vertically with 8 km/s, it will go up to a maximum height and then fall back to the earth. Eventually all of the objects KE is converted to PE and the object stops momentarily. It will then fall back to earth. If you send it up with the larger velocity of 11.2 km/s, the object will not come back (and thus will not come 'down'). At 11.2 km/s the object has more kinetic energy than the gravitational potential energy of the object at an infinite distance from the earth. This speed of 11.2 km/s is called **'escape speed'** of the earth. Escape speed obviously depends on the gravity of the the planet the object is leaving. For example, the 'escape speed' from the sun's gravity is 620 km/s. Only a few objects (Pioneer 10 and Voyager I and II) have ever gone this fast

Kepler's Laws of Orbital Motion

Before Newton, Johannes Kepler using the observations of Tycho Brahe had worked out empirical relationships for the motion of the (then) known planets. Understanding the motion of the planets had been a problem in astronomy since ancient times. Kepler's law can be stated as:

1. Planets follow elliptical orbits, with the sun at one focus of the ellipse. (Ellipses are oval shaped closed curves ... a squashed circle.) Note that will the orbits are ellipse, most of the planets, including earth have only slightly elliptical orbits i.e., their orbits are almost to circles

2. As a planet moves in its orbit, a line between the planet and the sun sweeps out equal areas in equal times.

3. The period, T, of a planet is proportional to its mean (average) distance from the sun to the $\frac{3}{2}$ power. As a formula this can be stated as $T \propto r^{\frac{3}{2}}$

Kepler's third law is the most useful of Kepler's three laws. It follows directly from Newton's law of gravity. Gravity is what provides the centripetal force to keep the planet in orbit around the sun:

$$F_{gravity} = \text{Centripetal Force}$$

$$G\frac{m_s m_p}{r^2} = m_p\frac{v^2}{r}$$

Since $v = \frac{2\pi r}{T}$ we have:

$$G\frac{m_s}{r^2} = \frac{(2\pi r/T)^2}{r}$$

$$G\frac{m_s}{r^2} = \frac{4\pi^2 r}{T^2}$$

$$T^2 = \frac{4\pi^2 r^3}{Gm_s}$$

$$T = \frac{2\pi}{\sqrt{Gm_s}} \cdot r^{\frac{3}{2}}$$

Which is Kepler's 3^{rd} law as an equality with the constant of proportionality.

Special Relativity

We normally think of living in a three dimensional universe. Objects have height, width and depth. To specify a location you have to give three coordinates, e.g., (east, north, altitude) or (x, y, and z). We normally forget about time. If you look at a star in the night sky, you can specify its location but you are looking at the star as it was many years ago. The light from the star left the star years ago. To correctly describe the star you also have to specify the time. What we see now is how the star was long ago. To completely describe an object we need to use the normal three dimensions **and time**. When we use this four dimensional description for an object, we call it **spacetime**. In relativity, we have to consider time along with the spatial dimensions.

Inertia Frames

If you were in a spaceship moving with constant velocity (no acceleration) and could not see the outside, you would have no way of knowing you were moving or not. Such a situation is called an inertial frame of reference. The laws of physics are the same if you are moving with constant velocity (non-accelerated) or 'fixed'.

Relativity is based on two postulates (assumptions). Everything that follows is based on these two ideas. The simplicity of these ideas and their far-reaching consequences is what make relativity such a 'beautiful' theory.

- All the laws of nature are the same in **all** inertial frames.
- The speed of light in free space has the same value (c = 3 x 10^8 m/s) in all inertial frames. The motion of the source and observer does not matter.

The first idea is very basic. You would not expect the laws of physics to change if you are in a car moving with constant velocity. The second is **counter-intuitive** to our normal experience. If you are moving in a car and throw a ball, a person catching the ball at the side of the road would catch the ball with a velocity that is the sum of the car's velocity plus the velocity of the ball. With light, it does not work like this. Shine a light from the star-ship Enterprise moving at $\frac{1}{2}$c and the receiver of the light will measure the light's speed at c (**not 1.5c !**)

Time Dilation

One consequence of the speed of light being the same in all inertia reference frames is that time is not measured the same. Because the speed of light must be a constant in all frames, time gets 'stretched' out in a moving inertial frame.

Suppose I am in inertial frame 1 and you are in inertial frame 2 moving with a velocity v. I measure some time (t_o = 1 minute say) with a clock. What time will I see elapse on the clock you have in frame 2 during the time t_o? The time in frame two is given by:

$$\mathrm{t} = \frac{t_o}{\sqrt{1-\frac{v^2}{c^2}}}$$

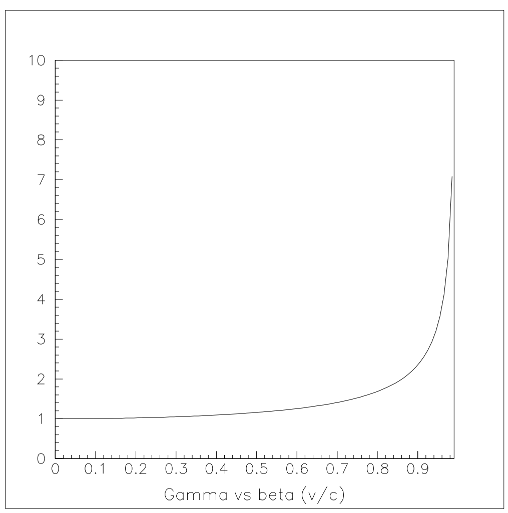
The time t_o is called the proper time. It is the time I measure in my own frame.

Let's look at the above equation. The factor $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, sometimes called a γ factor has some interesting properties. At any speed in the everyday world, this factor is 1. Because c is so very large, dividing a normal speed by c gives a very small number. Square that ratio of speeds and you get an even smaller number. Now subtract it from one and you have almost exactly 1. Take the square root and you are even closer to one.

The γ factor only becomes significant when you get near the speed of light.

speed (m/s)	% percent c	γ
1.50 x 10^8	50% c	1.15
2.25 x 10^8	75% c	1.51
2.40 x 10^8	80% c	1.67
2.61 x 10^8	87% c	2.00
2.70 x 10^8	90% c	2,29
2.85 x 10^8	95% c	3.20
2.97 x 10^8	99% c	7.09

If you are moving at $\frac{1}{2}$ the speed of light your clock will appear to run slow by 1.15 to an 'fixed' observer. The plot below shows the γ factor. When the ratio of v/c (the ratio of the speed over the speed of light) is small i.e. normal speed the γ factor is very nearly one. Only at 87% of the speed of light does the γ factor become 2.



Length Contraction

Space (length) also changes. This effect is normally called **Lorentz** contraction after Hendrick Lorentz. He had proposed such a contract to account for the negative result of the Michelson and Morley. (He was the same person who thought physics had all been worked out!). The relationship between length in different inertial frames is:

$$L = L_o \sqrt{1 - \frac{v^2}{c^2}}$$

A meter stick moving at 87% the speed of light with respect to you will appear to be on 50 cm long.

Momentum and Mass

If you apply a force to something the momentum (mass x velocity) changes. If the mass doesn't change, the speed has to increase. We know that something can not go faster than the speed of light.
The mass has to change! In special relativity the momentum is:

$$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We normally think of the mass as increasing with speed so that:

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_o is the 'rest mass' i.e., the mass in the observers inertial frame and m is the mass in the inertial frame moving with velocity v.

All of these effects may sound a bit esoteric. How could anyone really observe these effects? They happen all the time in sub-atomic particle accelerators. Particles that decay with a know life time live longer when they are moving a speeds near c. In order to work, particle accelerators have to allow for the increase in mass as they accelerate the particles to speed near the speed of light.

Equivalence of Mass and Energy

Einstein's special theory of relativity explained more than why there needs to be no 'ether'. It also explains where the stars get their energy. They convert mass to energy. It is probably the most famous equation of all time Yes, it is the famous:

$$E = mc^2$$

When we talked about mass increasing as a particle nears the speed of light, we could interpret the kinetic energy of the particle being mass. It works the other way also. If you can get a little bit of mass to convert to energy, you have a lot of energy since c^2 is a big number (9 x 10^{16}). One milligram of matter (a few grains of salt) would give you 9 x 10^{10} Joules. That is about the output of a large power plant for a couple of minutes!

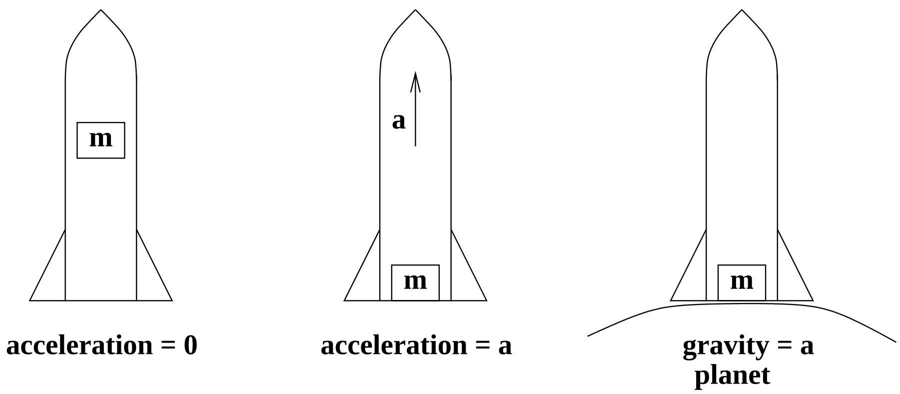
Again this really can be observed. Sub-atomic particles decay (change to other particles) to less massive particles. The excess mass is converted to energy.

The reverse process is '**Pair production**' where a γ ray (energy) converts to an electron and a positron (matter). We will talk more about this and other aspects of sub-atomic particles later.

General Relativity

The **special theory of relativity** is called 'special' because it deals with the special case of reference frames moving with constant velocity (inertial frames). After 1905, Einstein expanded relativity to cases where the reference frame is accelerated. In 1915 Einstein published his **general theory of relativity** which deals with accelerated reference frames. Essentially, general relativity turns out to be a new theory of gravity. It reproduces the results of Newton's law of gravity but can also make predictions where Newton's law of gravity breaks down.

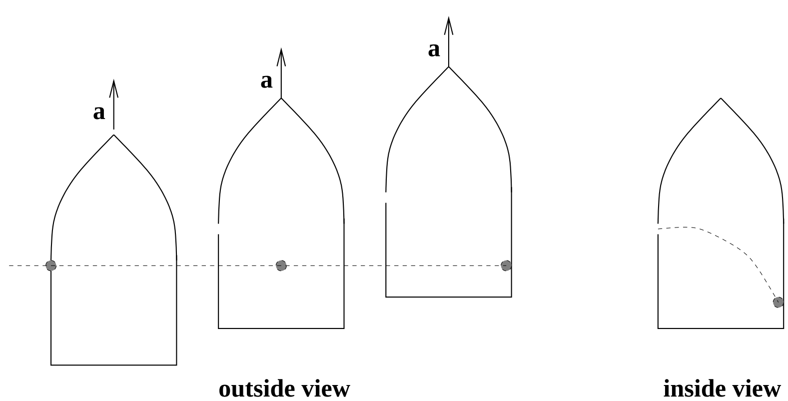
Consider a spaceship far from any planet or star. There are no windows in the spaceship. A mass will float around the spaceship cabin because there is no gravity. If the rocket motors are turned on, the mass will fall to the bottom of the ship. Likewise if the ship is on a planet, the mass will fall to the bottom of the ship.



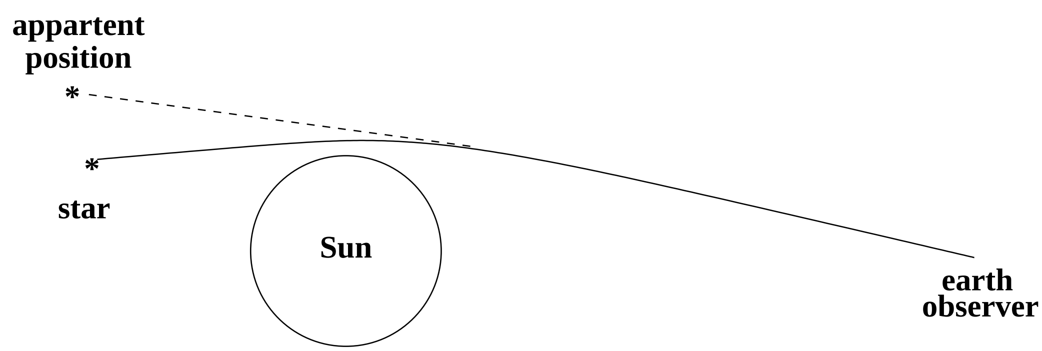
Unless you look outside and observe the acceleration of the ship you can not tell the difference between an acceleration and gravity. This is called the **principle of equivalence**:

- Local observation made in an accelerated frame of reference cannot be distinguished from observations made in a Newtonian gravitational field.

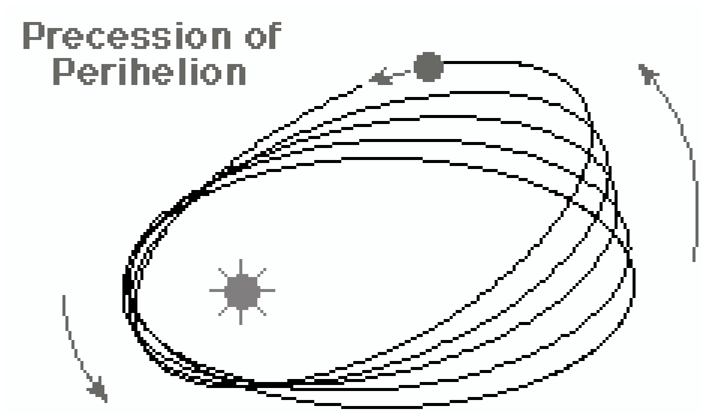
Consider a ball being thrown into the ship as it passes by with accelerated motion. From the outside, the ball travels in a straight line. To an observer on the inside of the windowless ship, the path of the ball is curved just like the path of the ball would be if the spaceship were setting in a gravitational field on the surface of a planet.



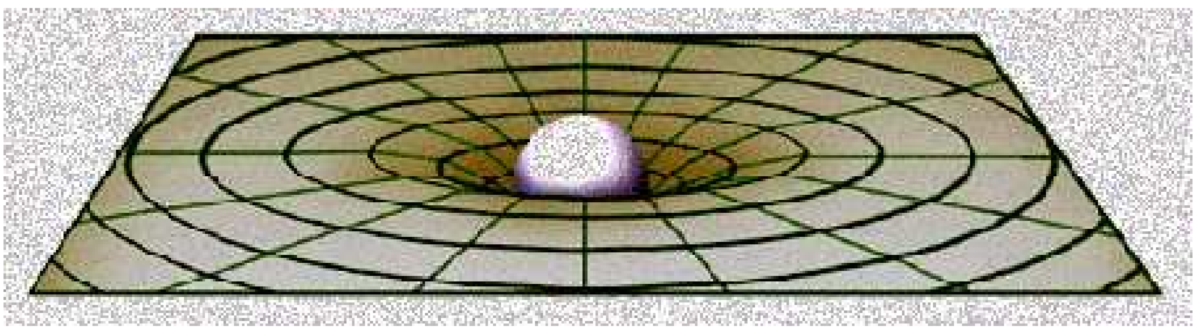
One consequence of this is that light bends in a gravitational field. If the ball were a beam of light, the observer in the ship would see the light beam bend. The bend is only a tiny amount since light travels so fast but the argument still applies. Since gravity and acceleration can not be distinguished by the principle of equivalence, light must bend in a gravitational field. In 1919, observations were made of star positions near the edge of the sun during a total eclipse. The apparent position of the star is shifted from its real position when the light passes near the disk of the Sun. The experiment has to be done during a total eclipse since the sun obscures the light from the star under normal conditions. The results of the experiment were in perfect agreement with the theory of general relativity.



Another supporting experiment which confirmed general relativity was the precession of the orbit of the planet Mercury. The orbit of Mercury is an ellipse. The angle of the axis of the ellipse rotates over time. The is call the **precession of the perihelion** of Mercury. The planet Vulcan had even been proposed to explain this observation in the late 1800.



Einstein's theory of general relativity tells us that a mass warps the **four dimensional space time** around the object. When this happens, **Euclidean geometry** no longer applies. A straight line is no longer the shortest path between two points. This is not as weird as it sounds. You are probably familiar with the idea of an airplane flying a 'great circle route'. A straight line on a map is not the shortest distance between Europe and New York because the earth is a sphere. Light takes the shortest distance between two space-time points in four dimensional space time. These shortest paths are called geodesics.



This explains how the universe can be finite, if very large, and you don't find a brick wall that says 'end of the universe'. The mass in the universe curves space and time back on itself. It is analogous to getting in a plane and flying in a straight line. Eventually, you go all the way around the earth and come back to where you started. Of course the earth is a two dimensional surface where as in general relativity, the universe is a four dimensional system.