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HOMEWORK #9#

Course 314,Introduction In Quantum Mechanics,Professor K. Griffioen

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1 Problem 10.4

(a) Use Equation 10.46 to calculate the geometric phase change when the infinite square well expands adiabatically from width ω_1 to width ω_2. Comment on this result.

(b) If the expansion occurs at a constant rate ($d\omega/dt = v$), what is the dynamic phase change for this process?

(c) If the well now contracts back to its original size, what is Berry's phase for the cycle?

2 Solution

(a) From equation 10.46, the expression of the geometric phase, and the general wave function are given by:

$$\begin{aligned} \gamma_n(t) = \quad & i\int_{\omega_1}^{\omega_2} < \psi_n|\frac{\partial\psi_n}{\partial\omega} > d\omega, \\ \psi_n = \quad & \sqrt{\frac{2}{\omega}} sin\frac{n\pi x}{\omega} \end{aligned} \tag{1}$$

It might be confussing the ω notation, but it represent the lenth of the square well potential, and not a frequency. We pursue to compute the scalar product:

$$\begin{aligned} \frac{\partial\psi_n}{\partial\omega} = \quad & -\frac{\sqrt{2}}{2}\frac{1}{\omega^{3/2}} sin\frac{n\pi x}{\omega} - \frac{n\pi x\sqrt{2}}{\omega^{5/2}} cos\frac{n\pi x}{\omega}, \\ < \psi_n|\frac{\partial\psi_n}{\partial\omega} >= \quad & -\frac{1}{\omega^2}\int_0^{\omega}\left(sin^2\frac{n\pi x}{\omega} + \frac{2n\pi x}{\omega} cos\frac{n\pi x}{\omega} sin\frac{n\pi x}{\omega}\right)dx, \end{aligned} \tag{2}$$

$$
\begin{aligned}
&= -\frac{1}{(n\pi)\omega}\int_0^{n\pi}\left(sin^2\beta + 2\beta sin\beta cos\beta\right)d\beta, \\
&= -\frac{1}{(n\pi)\omega}\left[-(n\pi)cos(n\pi)^2 + (n\pi)\right], \\
&= 0, \\
\gamma_n(t) &= 0
\end{aligned}
$$

Where we used the change of variable $\beta = \frac{n\pi x}{\omega}$, and n an integer. Then from (1) we obtained that the geometric phase does not change if we expand adiabatically the width of square well, it means that the phase does not depend of the width.

(b) The dynamic phase is given as:

$$\theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(t')dt' \tag{3}$$

Where the Bohr energy is given by :

$$E_n(t) = \frac{n^2\pi^2\hbar^2}{2m\omega_n^2(t)} \tag{4}$$

The expansion occurs at the constant rate $(d\omega/dt = v)$, so the variation in time of the well's width will be:

$$\psi_n(t) = \psi_o + vt \tag{5}$$

We plug (4) and (5) into (2) and we'll get the dynamic phase variation in time with respect of constant rate of variation of the width.

$$
\begin{aligned}
\theta_n(t) &= -\frac{1}{\hbar}\int_0^t \frac{n^2\pi^2\hbar^2}{2m(\omega_0 + vt')^2}dt', \qquad (6)\\
\theta_n(t) &= -\frac{n^2\pi^2\hbar^2}{2m\omega_0(\omega_0 + vt)}t
\end{aligned}
$$

(c) As we have shown in the part (a) the phase does not depend of the fact we expand or contract the width of well, so the value of Berry's phase is zero.

3 Problem 10.5

The delta-function well (Equation 2.96) supports a single bound state (Equation 2.111). Calculate the geometric phase change when α gradually increases from α_1 to α_2. If the increase occurs at a constant rate ($d\alpha/dt = c$), what is the dynamic phase change for this process?

4 Solution

From the equation 2.111 the wave function can be written in function of α as follows :

$$\begin{aligned} \psi(x) = & \quad \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}, \\ \gamma_n(t) = & \quad i\int_{\alpha_1}^{\alpha_2} < \psi(x)|\frac{\partial \psi}{\partial \alpha} > d\alpha, \\ E_\alpha = & \quad -\frac{m\alpha^2}{2\hbar^2} \end{aligned} \tag{7}$$

We compute the scalar product $< \psi(x)|\frac{\partial\psi}{\partial\alpha} >$ and we obtain:

$$\begin{aligned} < \psi(x)|\frac{\partial \psi}{\partial \alpha} >= & \quad \frac{m}{\hbar^2}\int_{-\infty}^{\infty} \sqrt{\alpha} e^{-m\alpha|x|/\hbar}\left[\frac{\partial}{\partial \alpha}\left(\sqrt{\alpha} e^{-m\alpha|x|/\hbar}\right)\right] dx, \\ = & \quad \frac{m}{\hbar^2}\int_{-\infty}^{\infty} \sqrt{\alpha} e^{-2m\alpha|x|/\hbar}\left[\frac{1}{2\sqrt{\alpha}} - \sqrt{\alpha}\left(\frac{m||x|}{\hbar^2}\right)\right], \\ = & \quad \frac{m}{2\hbar^2}\int_{-\infty}^{\infty} e^{-\frac{2m\alpha}{\hbar}|x|} dx - \frac{m^2\alpha}{\hbar^4}\int_{-\infty}^{\infty} |x| e^{-\frac{2m\alpha}{\hbar}|x|} dx, \\ = & \quad \frac{1}{2\alpha} - \frac{1}{2\alpha} = 0 \end{aligned} \tag{8}$$

So then we conclude that the geometric phase does not change with “strength” of delta function:

$$\gamma_n(t) = 0 \tag{9}$$

If the “strength” of delta function increases at the constant rate ($c = \frac{d\alpha}{dt}$), the dynamic phase changes as follows:

$$
\begin{aligned}
\theta_n(t) = & \quad -\frac{1}{\hbar}\int_0^t E_n(t')dt', \\
c = & \quad \frac{d\alpha}{dt} \longrightarrow \alpha(t) = \alpha_0 + ct, \\
E_n(t) = & \quad -\frac{m}{2\hbar}\alpha^2(t) = -\frac{m}{2\hbar}(\alpha_o + ct)^2, \\
\theta_n(t) = & \quad \frac{m}{2\hbar^3}\int_0^t (\alpha_0 + ct)^2 dt = \frac{m}{2\hbar^3}\left(\alpha_0^2 t + c\alpha_0 t^2 + c^2\frac{t^3}{3}\right)
\end{aligned} \tag{10}
$$

5 Problem 11.2

Construct the analogs to Equation 11.12 for one-dimensional and two-dimensional scattering.

6 Solution

From equation 11.12 the general form of *Shrödinger* equation has an approximative form:

$$
\psi(r,\theta) \approx A\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right], k = \frac{\sqrt{2mE}}{\hbar}, \qquad r \to \infty \tag{11}
$$

The above equation represent the spatial scattering wave function, where $|r| = \sqrt{x^2+y^2+z^2}$, and depends of symmetry of the potential(azimuthal, spherical,cylindrical). In one-dimension the equation 11.12 takes the following form:

$$
\psi(x) = Ae^{ikx} + Be^{-ikx} \tag{12}
$$

Where the scattering amplitude $f(\theta)$ is a constant(B), and the probability of scattering in the x direction is conserved. For the two-dimensional scattering the above equation takes the following form:

$$
\psi(r,\theta) = A\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{\sqrt{r}}\right] \tag{13}
$$

Also the probability of scattering in the θ direction is conserved by choosing the form of outgoing (scattered) wave of form $\psi_{out} = f(\theta)\frac{e^{ikr}}{\sqrt{r}}$:

$$
dP = |\psi_{inc}|^2 dS = |\psi_{scatt}|^2 dS = \int \psi^*\psi r dr d\theta \tag{14}
$$

References

[1] D.J.Griffiths, Introduction to Quantum Mechanics, 1995