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HOMEWORK #8#

Course 314,Introduction In Quantum Mechanics,Professor K. Griffioen

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1 Problem 9.3

Suppose the perturbation takes the form of a delta function (in time) :

$$H' = U\delta(t - t_0) \tag{1}$$

assume that $U_{aa} = U_{bb} = 0$, and let $U_{ab} = \alpha$. If $c_a(-\infty) = 1$ and $c_b(-\infty) = 0$, find $c_a(t)$ and $c_b(t)$, and check that $|c_a(t)|^2 + |c_b(t)|^2 = 1$. What is the probability ($P_{a\to b}$) that a transition occurs?

2 Solution

From relations 9.13 we have that the coefficients $\dot{c}_a$ and $\dot{c}_b$ are the following :

$$\begin{aligned} \dot{c}_a = & \quad -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b, \\ \dot{c}_b = & \quad -\frac{i}{\hbar} H'_{ba} e^{1\omega_0 t} c_a \end{aligned} \tag{2}$$

If we'll plug the expression of H' into the above relations we'll obtain the following :

$$\begin{aligned} \dot{c}_a = & \quad -\frac{i}{\hbar} \alpha\delta(t - t_0) e^{-i\omega_0 t} c_b, \\ \dot{c}_b = & \quad -\frac{i}{\hbar} \alpha^* \delta(t - t_0) e^{i\omega_0 t} c_a \end{aligned} \tag{3}$$

Integrating those relations we'll get :

$$\begin{aligned}\int_{-\infty}^{t} \dot{c}_a dt' = & \int_{-\infty}^{t} -\frac{i}{\hbar}\alpha\delta(t'-t_0)e^{-i\omega_0 t'}c_b(t')dt', \\ \int_{-\infty}^{t} \dot{c}_b dt' = & \int_{-\infty}^{t} -\frac{i}{\hbar}\alpha^*\delta(t'-t_0)e^{i\omega_0 t'}c_a(t')dt'\end{aligned} \tag{4}$$

For $t < t_0$ we'll obtain that :

$$\begin{aligned}c_a(t) = & \quad 1, \\ c_b(t) = & \quad 0\end{aligned} \tag{5}$$

For $t \geq t_0$ we'll obtain that :

$$\begin{aligned}c_a(t) = & \quad 1 - \frac{i}{\hbar}\alpha e^{-i\omega_0 t_0}c_b(t_0), \\ c_b(t) = & \quad -\frac{i}{\hbar}\alpha^* e^{i\omega_0 t_0}c_a(t_0)\end{aligned} \tag{6}$$

For $t < t_0$ the probability that a transition occurs ($P_{a\to b}$) is given by :

$$P_{a\to b} = |c_b|^2 = 0 \tag{7}$$

For the case $t \geq t_0$ we'll use the step function as follows(see Problem 2.24):
(i) we'll take $c_a(t_0)$ to be half way between 1 and $\left[1 - \frac{i}{\hbar}\alpha e^{-i\omega_0 t-0}c_b(t_0)\right]$
(ii) we'll take $c_b(t_0)$ to be half way between 0 and $\left[-\frac{i}{\hbar}\alpha^* e^{i\omega_0 t_0}c_a(t_0)\right]$.
yWe'll obtain then :

$$\begin{aligned}c_a(t_0) = & \quad 1 - \frac{i}{\hbar}\frac{\alpha}{2}e^{-i\omega_0 t_0}c_{t_0}, \\ c_b(t_0) = & \quad -\frac{i}{\hbar}\frac{\alpha^*}{2}e^{i\omega_0 t_0}c_a(t_0),\end{aligned} \tag{8}$$

$$
\begin{aligned}
c_a(t_0) = & \quad \frac{1}{1+\left(\frac{i}{\hbar}\frac{\alpha}{2}e^{-i\omega_0 t_0}\right)\left(-\frac{i}{\hbar}\frac{\alpha^*}{2}e^{i\omega_0 t_0}\right)}, \\
c_b(t_0) = & \quad \frac{-\frac{i}{\hbar}\frac{\alpha^*}{2}e^{i\omega_0 t_0}}{1+\left(\frac{i}{\hbar}\frac{\alpha}{2}e^{-i\omega_0 t_0}\right)\left(-\frac{i}{\hbar}\frac{\alpha^*}{2}e^{i\omega_0 t_0}\right)}
\end{aligned}
$$

If we plug into (6) then the time dependent coefficients, for $t \geq t_0$ will be :

$$
\begin{aligned}
c_a(t) = & \quad 1 - \frac{\frac{i}{\hbar}\alpha e^{-i\omega_0 t_0}\left(-\frac{i}{\hbar}\frac{\alpha^*}{2}\right)e^{i\omega_0 t_0}}{1+\frac{|\alpha|^2}{4\hbar^2}}, \\
c_b(t) = & \quad \frac{-\frac{i}{\hbar}\alpha^* e^{i\omega_0 t_0}}{1+\frac{|\alpha|^2}{4\hbar^2}}
\end{aligned} \tag{9}
$$

The probability that the transition occurs ($P_{a\to b}$), for $t \geq t_0$ will be :

$$
\begin{aligned}
P_{a\to b} = |c_b|^2 = & \quad \frac{|\alpha|^2/\hbar^2}{(1+|\alpha|^2/4\hbar^2)^2}, \\
|c_a|^2 + |c_b|^2 = & \quad 1
\end{aligned} \tag{10}
$$

3 Problem 9.10

Calculate the lifetime (in seconds) for each of the four $n = 2$ states of hidrogen.*Hint*: You'll need to evaluate matrix elements of the form $< \psi_{100}|x|\psi_{200} >, < \psi_{100}|x|\psi_{211} >$, and so on. Remember that $x = rsin\theta cos\phi, y = rsin\theta sin\phi$, and $z = rcos\theta$. Most of integrals are zero, so scan them before you start calculating.

4 Solution

For $n = 2$ we have the following wave functions: $\psi_{200}, \psi_{21,-1}, \psi_{210}, \psi_{211}$. The transition rate is give by :

$$
A = \quad \frac{\omega^3|\vec{p}|^2}{3\pi\epsilon_0\hbar c^3}, \tau = \frac{1}{A}, \tag{11}
$$

$$A = \frac{4|\vec{p}|^2(\hbar\omega)^3}{12\pi\epsilon_0(\hbar c)^3\hbar},$$
$$\vec{p} = -e < n'l'm'|\vec{r}|nlm >$$

We use the well known constants $\hbar c = 197 MeV - fm$,and $\frac{e^2}{4\pi\epsilon_0} = 1.44 MeV - fm$ in computting the transition rate A :

$$A = 4.05 \times 10^{-10}| < n'l'm'|\vec{r}|nlm > |^2 \left(\frac{1}{(ns)(fm)^2}\right) \quad (12)$$

We ll use the spherical polar coordinates to exprim the $\vec{r}$, and the angular functions Y_{lm} to compute the matrix elements $< n'l'm'|\vec{r}|nlm >$:

$$\vec{r} = rsin\theta cos\phi\hat{i} + rsin\theta sin\phi\hat{j} + rcos\theta\hat{k}, \quad (13)$$
$$cos\theta = \sqrt{\frac{4\pi}{3}}Y_{10},$$
$$sin\theta cos\phi = (Y_{11} + Y_{1,-1})\sqrt{\frac{2\pi}{3}},$$
$$sin\theta sin\phi = (-Y_{11} - Y_{1,-1})\frac{1}{i}\sqrt{\frac{2\pi}{3}}$$

The wave functions are :

$$\psi_{200} = R_{20}Y_{00}, Y_{00} = \frac{1}{\sqrt{4\pi}}, \quad (14)$$
$$\psi_{21,-1} = R_{21}Y_{1,-1}, Y_{1,-1} = \sqrt{\frac{3}{8\pi}}sin\theta e^{-i\phi},$$
$$\psi_{210} = R_{21}Y_{10}, Y_{10} = \sqrt{\frac{3}{4\pi}}cos\theta,$$
$$\psi_{21,-1} = R_{21}Y_{11}, Y_{11} = -\sqrt{\frac{3}{8\pi}}sin\theta e^{i\phi}$$

The matrix elements which are not equal with zero are :

$$< 210|z|100 >, \quad < 211|y|100 >, \quad < 21-1|y|100 >, \quad (15)$$

$$< 211|x|100 >, \quad < 21-1|x|100 >,$$

Each of these integrals has the factor :

$$\int_0^\infty r^3 R_{21} R_{10} dr = a \frac{2^8}{3^4\sqrt{6}} \tag{16}$$

The angular contribution in each direction(x, y, z) will be :

$$\begin{aligned} x : &\quad \longrightarrow \tfrac{1}{3}, \\ y : &\quad \longrightarrow \tfrac{1}{6}, \\ z : &\quad \longrightarrow \tfrac{1}{6} \end{aligned} \tag{17}$$

Since $|\vec{p}|^2 = | < x > |^2 + | < y > |^2 + | < z > |^2$ each transition gets a factor (1/3).Then we plug the above results into () and will obtain :

$$\begin{aligned} | < n'l'm'|\vec{r}|nlm > |^2 = &\quad a^2 \frac{2^{15}}{3^{10}}, \\ A = &\quad 0.629(ns)^{-1}, \\ \tau = &\quad 1/A = 1.59ns \end{aligned} \tag{18}$$

No transitions have place between ψ_{200} and ψ_{100}, so in this case $\tau \longrightarrow \infty$.

5 Problem 9.13

An electron in the $n = 3, l = 0, m = 0$ state of hidrogen decays by a sequence of (electric dipole) transitions to the ground state.

(a) What decays routes are open to it ? Specify them in the followingway ;

$$|300 > \to |nlm > \to |n'l'm' > \to ... \to |100 > . \tag{19}$$

(b) If you had a bottle ful of atoms in this state, what fraction of them would decay via each route ?

(c) What is the lifetime of this state?

6 Solution

(a) As we saw in the previous problem (9.10) the decays $|300> \to |21m>$ have the same angular integrals as the decays $|21m> \to |100>$. Only the radial integrals are different, but they are the same for each value of m :

$$
\begin{aligned}
|300> &\longrightarrow |210> \longrightarrow |100>, \\
|300> &\longrightarrow |21-1> \longrightarrow |100>, \\
|300> &\longrightarrow |211> \longrightarrow |100>
\end{aligned} \tag{20}
$$

(b) All the above states (20) are equally likely.

(c) We need to evaluate the integral $\int_0^\infty r^3 R_{30} R_{21} dr$:

$$
\begin{aligned}
\int_0^\infty r^3 R_{30} R_{21} dr &= \int_0^\infty r^3 \frac{2}{\sqrt{27}} \left[1 - \frac{2r}{3a} + \frac{2}{27}\left(\frac{r}{a}\right)^2\right] e^{-r/3a} \frac{1}{\sqrt{24}} \frac{1}{a^{3/2}} \frac{r}{a} e^{-r/2a} dr \\
&= \frac{3456}{15625} a\sqrt{18}
\end{aligned} \tag{21}
$$

The transition rate (A) will be the same for each state ($|210>, |21-1>, |211>$), and will be 0.7273 of the transition rate obtained in problem 9.10, so the lifetime of the state $|300>$ will be :

$$
\begin{aligned}
A &= 0.457(ns)^{-1}, \\
\tau &= \frac{1}{3 \times A} = 0.73ns
\end{aligned} \tag{22}
$$

References

[1] D.J.Griffiths,Introduction To Quantum Mechanics,1995