1 Problem 7.17

The fundamental problem in harnessing nuclear fusion is getting the two particles (say, two deuterons) close enough together for the attractive (but short-range) nuclear force to overcame the Coulomb repulsion. The "brute force" method is to heat the particles to fantastic temperatures and allow teh random collisions to bring them together. A more exotic proposal is muoncatalysis, in which we construct a "hydrogen molecule ion", only with deuterons in place of protons, and a muon in place of the electron. Predict the equilibrium separation distance between the deuterons in such a structure, and explain why muons are superior to electrons for this purpose.

2 Solution

The raport between atomic masses of moun and electron is :

\[
\frac{m_\mu}{m_e} \approx 207
\]  (1)

The point of minimum energy has \( x = 2.4 \) and is given by \( x = \frac{R}{a} \) where \( R \) is the separation of the two nuclei, and the Bohr radius is given by :

\[
a = \frac{4\pi\varepsilon k^2}{m_e^2}
\]  (2)

We simly replace the mass of the electron with the mass of the muon :

\[
R = a_\mu x = \frac{a_\mu}{a_e} (a_e x) = \frac{m_e}{m_\mu} (1.27 \times 10^{-10}) m = 6.14 \times 10^{-13} m
\]  (3)
3 Problem 8.7

Use the WKB approximation to find the allowed energies of the harmonic oscillator.

4 Solution

The potential (perturbation) of the harmonic oscillator is $E = \frac{1}{2}m\omega^2x_0^2$, and using the WKB approximation we have:

$$p(x) = \sqrt{2m(E - \frac{1}{2}m\omega^2x_0^2)},$$

$$\int_{-x_0}^{x_0} p(x)dx = \left(n - \frac{1}{2}\right)\pi\hbar, n = 1, 2, 3...$$

$$\int_{-x_0}^{x_0} \sqrt{2m\omega^2} \sqrt{1 - \left(\frac{x}{x_0}\right)^2} \frac{1}{2}m\omega^2 dx = \int_{-1}^{1} m\omega x_0^2\sqrt{1 - u^2}du,

\frac{1}{2}m\omega x_0^2\pi = \left(n - \frac{1}{2}\right)\pi\hbar,$$

$$E = \left(n - \frac{1}{2}\right)\hbar\omega$$

where we changed variables $u = \frac{x}{x_0}$. The result is exactly the harmonic oscillator result, since $n$ starts with 1 in this case, instead of with 0.

5 Problem 8.13

For spherically symmetrical potentials, we apply the WKB approximation to the radial equation, (Equation 4.37). In the case $l = 0$, it is reasonable to use Equation 8.47 in the form:

$$\int_{0}^{r_0} p(r)dr = (n - 1/4)\pi\hbar$$

where $r_0$ is the turning point (in effect, we treat $r = 0$ as an infinite wall). Apply this formula to estimate the allowed energies of a particle in the logarithmic potential:

$$V(r) = V_0ln(r/a)$$
for constants $V_0$ and $a$). Treat only the case $l = 0$. Show that the spacing between the levels is independent of mass.

6 Solution

From WKB approximation we have:

$$\int_0^{r_n} = (n - 1/4)\pi h$$  \hspace{1cm} (7)

The $n$ and $(n + 1)$ energy levels will be given by:

$$E_{n+1} = V_0 ln \frac{r_{n+1}}{a},$$

$$E_n = V_0 ln \frac{r_n}{a},$$

$$E_{n+1} - E_n = V_0 \left( ln \frac{r_{n+1}}{a} - ln \frac{r_n}{a} \right),$$

$$E_{n+1} - E_n = V_0 \frac{r_{n+1}}{r_n}$$  \hspace{1cm} (9)

We’ll plug into (7) and we’ll obtain:

$$ln \int_0^{r_n} \sqrt{2m \left( V_0 ln \frac{r_n}{a} - V_0 ln \frac{r}{a} \right)} dr = ln \left[ \left( n - \frac{1}{4} \right)\pi h \right],$$

$$ln \sqrt{2mV_0} + ln \int_0^{r_n} \sqrt{ln(r_n/r)}dr = ln\pi h + ln(n - 1/4),$$

$$ln\sqrt{2mV_0} + ln \int_0^{r_n+1} \sqrt{ln(r_{n+1}/r)}dr = ln\pi h + ln(n + 3/4)$$

We’ll substract the above relations one from each other and we’ll make the following change of variable:

$$u = \frac{r}{r_n}, du = \frac{dr}{r_n},$$  \hspace{1cm} (11)
\[ u = \frac{r}{r_{n+1}}, du = \frac{dr}{r_{n+1}} \]

We'll obtain the relations:

\[
\begin{align*}
\ln \left( \frac{n + 3/4}{n - 1/4} \right) &= \ln r_{n+1} - \ln r_n, \\
\ln \frac{r_{n+1}}{r_n} &= \frac{1}{V_0} (E_{n+1} - E_n), \\
E_{n+1} - E_n &= V_0 \ln \left( \frac{n + 3/4}{n - 1/4} \right)
\end{align*}
\]

7 Problem 9.1

A hydrogen atom is placed in a (time-dependent) electric field \( E = E(t) \hat{k} \). Calculate all four matrix elements \( H'_{ij} \) of the perturbation \( H' = -eEz \) between the ground state \( (n = 1) \) and the (quadruply degenerate) first excited states \( (n = 2) \). Also show that \( H'_{ij} = 0 \) for all five states. Note: there is only one integral to be done here, if you exploit oddness with respect to \( z \). As a result, only one of the \( n = 2 \) states is "accessible" from the ground state by a perturbation of this form, and therefore the system functions as a two-level configuration - assuming transitions to higher excited states can be ignored.

8 Solution

\[
\begin{align*}
H' &= -eEz, z = r \cos \theta = \sqrt{\frac{4\pi}{3}} r Y^0_1, \\
Y'_{l'm'} &= \delta_{ll'} \delta_{mm'}
\end{align*}
\]

where \( Y^m_l \) are the angular functions of the wave functions and:

\[
\begin{align*}
\int Y_{l'm'}^{*} Y_{lm} d\omega &= \delta_{ll'} \delta_{mm'}, \\
Y_{00} &= 0
\end{align*}
\]

The wave functions are given by:

\[
\psi_{100} = Y_{00} R_{10},
\]
\[ \psi_{200} = Y_{00} R_{20}, \]
\[ \psi_{210} = Y_{10} R_{21}, \]
\[ \psi_{21,-1} = Y_{1,-1} R_{21}, \]
\[ \psi_{211} = Y_{11} R_{21} \]

The matrix elements of $H'$ are given by:

\[ H'_{ij} = \langle n'l'm'|H'|nlm \rangle = -eE < 100|z|2lm >= -eE Y_{00} Y_{10} Y_{lm} \]  

But from the definition of $\langle \psi|H|\psi \rangle$ we get that:

\[ \int d\Omega Y_{00} Y_{10} Y_{lm} = Y_{00} \int d\Omega Y_{10} Y_{lm} = Y_{00} \delta_{l1} \delta_{m0} \]  

And we can observe that only $\psi_{210}$ is possible. We plug $\psi_{210}$ into (16) and the nonvanishing term of $H'$ will be:

\[ < \psi_{210}|H'|\psi_{210} = -eE < R_{10}|r|R_{21} > \frac{1}{\sqrt{12\pi}} \]

\[ \int_0^\infty r^3 \frac{2}{a^{3/2}} e^{-r/a} \frac{1}{\sqrt{24}} \frac{1}{a^{3/2}} \frac{r}{a} e^{-r/2a} dr = \frac{2}{a^3} \frac{1}{\sqrt{24}} \left( \frac{2a}{3} \right)^5 \int_0^\infty u^4 e^{-u} du, \]

\[ < H' > = -eE a \frac{2! \sqrt{5}}{3^3} = -0.745eEa \]

References