April 26, 2000

HOMEWORK #7#

Course 314,Introduction In Quantum Mechanics,Professor K. Griffioen

Department of Physics, College of William and Mary, Williamsburg, Virginia 23185

1 Problem 7.17

The fundamental problem in harnessing nuclear fusion is getting the two particles (say, two deuterons) close enough together for the attractive (but short-range) nuclear force to overcame the Coulomb repulsion. The "brute force" method is to heat the particles to fantastic temperatures and allow teh random collisions to bring them together. A more exotic proposal is **muoncatalysis**,in which we construct a "hydrogen molecule ion", onlywith deuterons in place of protons, and a *muon* in place of the electron. Predict the equilibrium separation distance between the deuterons in such a structure, and explain why muons are superior to electrons for this purpose.

2 Solution

The raport between atomic masses of moun and electron is :

$$\frac{m_\mu}{m_e} \approx 207 \tag{1}$$

The point of minimum energy has $x = 2.4$ and is given by $x = \frac{R}{a}$ where R is the separation of the two nuclei, and the Bohr radius is given by :

$$a = \frac{4\pi\epsilon k^2}{me^2} \tag{2}$$

We simly replace the mass of the electron with the mass of the muon :

$$R = a_\mu x = \frac{a_\mu}{a_e}(a_e x) = \frac{m_e}{m_\mu}(1.27 \times 10^{-10})m = 6.14 \times 10^{-13}m \tag{3}$$

3 Problem 8.7

Use the WKB approximation to find the allowed energies of the harmonic oscillator.

4 Solution

The potential(perturbation) of the harmonic oscillator is $E = \frac{1}{2}m\omega^2 x_0^2$, and using the WKB approximation we have :

$$\begin{aligned} p(x) = & \sqrt{2m(E - \frac{1}{2}m\omega^2 x_0^2)}, \\ \int_{-x_0}^{x_0} p(x)dx = & \left(n - \frac{1}{2}\right)\pi\hbar, n = 1, 2, 3... \\ \int_{-x_0}^{x_0} \sqrt{2mx_0^2}\sqrt{1 - \left(\frac{x}{x_0}\right)^2}\sqrt{\frac{1}{2}m\omega^2}dx = & \int_{-1}^{1} m\omega x_0^2\sqrt{1 - u^2}du, \\ \frac{1}{2}m\omega x_0^2\pi = & \left(n - \frac{1}{2}\right)\pi\hbar, \\ E = & \left(n - \frac{1}{2}\right)\hbar\omega \end{aligned} \tag{4}$$

where we changed variables $u = \frac{x}{x_0}$.The result is exactly the harmonic oscillator result, since n starts with 1 in this case, instead of with 0.

5 Problem 8.13

For spherically symmetrical potentials, we apply the WKB approximation to the radial equation, (Equation 4.37). In the case $l = 0$, it is reasonable to use Equation 8.47 in the form :

$$\int_0^{r_0} p(r)dr = (n - 1/4)\pi\hbar \tag{5}$$

where r_0 is the turning point (in effect, we treat $r = 0$ as an infinite wall). Apply this formula to estimate the allowed energies of a particle in the logarithmic potential :

$$V(r) = V_0 ln(r/a) \tag{6}$$

(for constants V_0 and a). Treat only the case $l = 0$. Show that the spacing between the levels is independent of mass.

6 Solution

From WKB approximation we have :

$$\int_0^{r_n} = (n - 1/4)\pi\hbar \tag{7}$$

The n and $(n+1)$ energy levels will be given by :

$$\begin{aligned} E_{n+1} = & \quad V_0 ln\frac{r_{n+1}}{a}, \\ E_n = & \quad V_0 ln\frac{r_n}{a}, \\ E_{n+1} - E_n = & \quad V_0\left(ln\frac{r_{n+1}}{a} - ln\frac{r_n}{a}\right), \\ E_{n+1} - E_n = & \quad V_0 ln\frac{r_{n+1}}{r_n} \end{aligned} \tag{8}$$

$$\tag{9}$$

We'll plug into (7) and we'll obtain :

$$\begin{aligned} ln\int_0^{r_n}\sqrt{2m\left(V_0 ln\frac{r_n}{a} - V_0 ln\frac{r}{a}\right)}dr = & \quad ln\left[\left(n - \frac{1}{4}\right)\pi\hbar\right], \\ ln\int_0^{r_{n+1}}\sqrt{2m\left(V_0 ln\frac{r_{n+1}}{a} - V_0 ln\frac{r}{a}\right)}dr = & \quad ln\left[\left(n + \frac{3}{4}\right)\pi\hbar\right], \\ ln\sqrt{2mV_0} + ln\int_0^{r_n}\sqrt{ln(r_n/r)}dr = & \quad ln\pi\hbar + ln(n - 1/4), \\ ln\sqrt{2mV_0} + ln\int_0^{r_{n+1}}\sqrt{ln(r_{n+1}/r)}dr = & \quad ln\pi\hbar + ln(n + 3/4) \end{aligned} \tag{10}$$

We'll substract the above relations one from each other and we'll make the following change of variable:

$$u = \frac{r}{r_n}, du = \frac{dr}{r_n}, \tag{11}$$

$$u = \frac{r}{r_{n+1}}, du = \frac{dr}{r_{n+1}}$$

We'll obtain the relations :

$$\begin{aligned} ln\left(\frac{n+3/4}{n-1/4}\right) &= lnr_{n+1} - lnr_n, \\ ln\frac{r_{n+1}}{r_n} &= \frac{1}{V_0}(E_{n+1} - E_n), \\ E_{n+1} - E_n &= V_0 ln\left(\frac{n+3/4}{n-1/4}\right) \end{aligned} \quad (12)$$

7 Problem 9.1

A hidrogen atom is placed in a (time-dependent) electric field $E = E(t)\hat{k}$. Calculate all four matrix elements H'_{ij} of the perturbation $H' = -eEz$ between the ground state ($n = 1$) and the (quadruply degenerate) first excited states ($n = 2$). Also show that $H'_{ij} = 0$ for all five states.*Note*: there is only one integral to be done here, if you exploit oddness with respect to z. As a result, only one of the $n = 2$ states is "accsessible" from the ground state by a perturbation of this form, and therefore the system functions as a two-level configuration - assuming transitions to higher excited states can be ignored.

8 Solution

$$H' = -eEz, z = rcos\theta = \sqrt{\frac{4\pi}{3}} rY_1^0, \quad (13)$$

where Y_l^m are the angular functions of the wave functions and :

$$\begin{aligned} \int Y^*_{l'm'} Y_{lm} d\omega &= \delta_{ll'}\delta_{mm'}, \\ Y_{00} &= 0 \end{aligned} \quad (14)$$

The wave functions are given by :

$$\psi_{100} = Y_{00}R_{10}, \quad (15)$$

$$
\begin{aligned}
\psi_{200} = &\quad Y_{00}R_{20}, \\
\psi_{210} = &\quad Y_{10}R_{21}, \\
\psi_{21,-1} = &\quad Y_{1,-1}R_{21}, \\
\psi_{211} = &\quad Y_{11}R_{21}
\end{aligned}
$$

The matrix elements of H'are given by :

$$
H'_{ij} =< n'l'm'|H'|nlm >= -eE < 100|z|2lm >= -eEY_{00}Y_{10}Y_{lm} \tag{16}
$$

But from the definition of $< \psi|H|\psi >$ we get that :

$$
\int d\Omega Y_{00}Y_{10}Y_{lm} = Y_{00}\int d\Omega Y^*_{10}Y_{lm} = Y_{00}\delta_{l1}\delta_{m0} \tag{17}
$$

And we can observe that only ψ_{210} is possible.We plug ψ_{210} into (16) and the nonvanishing term of H' will be :

$$
\begin{aligned}
< \psi_{210}|H'|\psi_{210} = &\quad -eE < R_{10}|r|R_{21} > \frac{1}{\sqrt{12\pi}}, \qquad (18) \\
\int_0^\infty r^3\frac{2}{a^{3/2}}e^{-r/a}\frac{1}{\sqrt{24}}\frac{1}{a^{3/2}}\frac{r}{a}e^{-r/2a}dr = &\quad \frac{2}{a^4}\frac{1}{\sqrt{24}}\left(\frac{2a}{3}\right)^5\int_0^\infty u^4e^{-u}du, \\
< H' >= &\quad -eEa\frac{2^{15/2}}{3^5} = -0.745eEa
\end{aligned}
$$

References

[1] D.J.Griffiths,Introduction To Quantum Mechanics,1995