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**Homework #5#**

Course 314,Introduction In Quantum Mechanics,Professor K. Griffioen

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# 1 Problem 6.9

Consider a quantum system with just *three* linearly independent states. The Hamiltonian, in matrix form, is

$$H = V_0 \begin{pmatrix} (1-\epsilon) & 0 & 0 \\ 0 & 1 & \epsilon \\ 0 & \epsilon & 2 \end{pmatrix} \tag{1}$$

where $V_0$ is a constant and $\epsilon$ is some small number ($\epsilon << 1$).

**(a)** Write down the eigenvectors and eigenvalues of the *unperturbed* Hamiltonian ($\epsilon = 0$).

**(b)** Solve for the *exact* eigenvalues of $H$ . Expand each of them as a power series in $\epsilon$, up to second order.

**(c)** Use first- and second- order nondegenerate perturbation theory to find the aproximate eigenvalue for the state that grows out of the nondegenerate eigenvector of $H^0$. Compare the exact result from (b).

**(d)** Use *degenerate* perturbation theory to find the first - order correction to the two initially degenerate eigenvalues. Compare the exact results.

# 2 Solution

**(a)** The perturbed Hamiltonian H' and the unperturbed Hamiltonian $H^0$ will be :

$$\begin{aligned} H' &= V_0 \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix} \\ H^0 &= V_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{aligned} \tag{2}$$

The eigenvectors and the eigenvalues of $H^0$ are then :

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \rightarrow \beta_1 = 1 \tag{3}$$
$$\begin{pmatrix}0\\1\\0\end{pmatrix} \rightarrow \beta_2 = 1$$
$$\begin{pmatrix}0\\0\\1\end{pmatrix} \rightarrow \beta_3 = 2$$

**(b)** We'll solve the classical eigenvalues problem and we will have :

$$\begin{aligned}
\begin{vmatrix} (1-\epsilon)-\lambda & 0 & 0 \\ 0 & 1-\lambda & \epsilon \\ 0 & \epsilon & 2-\epsilon \end{vmatrix} &= 0, \\
(1-\epsilon-\lambda)(1-\lambda)(2-\lambda) - \epsilon^2(1-\epsilon-\lambda) &= 0, \\
1-\epsilon-\lambda &= 0 \longrightarrow \lambda_1 = 1-\epsilon \\
(1-\lambda)(2-\lambda) &= \epsilon^2 7 \\
\lambda^2 - 3\lambda + 2 - \epsilon &= 0, \longrightarrow \lambda_\pm = \frac{3}{2} \pm \frac{1}{2}\sqrt{1+4\epsilon^2}
\end{aligned} \tag{4}$$

If we expand in Taylor series the solutions $\lambda_\pm$ we obtain :

$$\begin{aligned}
\lambda_- &= 1-\epsilon^2 \\
\lambda_+ &= 2+\epsilon^2
\end{aligned} \tag{5}$$

where we used that :$(1 \pm 4\epsilon^2)^{1/2} = \frac{3}{2} \pm \frac{1}{2} \pm \epsilon^2 \mp \epsilon^4 \pm ....$

So the exact eigenvalues, up to second order are :

$$\begin{aligned}
\lambda_1 &= 1-\epsilon, \\
\lambda_2 &= 1-\epsilon^2,
\end{aligned} \tag{6}$$

$$\lambda_3 = \quad 2+\epsilon^2$$

y **(c)** In this case, $\lambda = 2$ and using the non-degenerate perturbation theory we'll obtain :

$$\begin{aligned}
E^0 &= \quad 2, \qquad (7)\\
E^1 &= \quad \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\epsilon & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0,\\
E^2 &= \quad \frac{1}{1-2}\left[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\epsilon & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]^2 + \frac{1}{1-2}\left[\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\epsilon & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]^2 = 2 + 0\epsilon + \epsilon^2 + \ldots
\end{aligned}$$

The result is the same with the result obtained from (b).

**(d)** Using the degenerate perturbation theory for the first - order corection for the initially degenerate eigenvalues we'll have ;

$$\begin{aligned}
\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} H' \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= \quad -\epsilon \qquad (8)\\
\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} H' \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} &= \quad 0\\
\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} H' \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= \quad 0\\
\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} H' \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} &= \quad 0
\end{aligned}$$

We resolve the eigenvalues problem and we'll find the first - order correction :

$$\begin{pmatrix} -\epsilon & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow \begin{vmatrix} -\epsilon - \lambda & 0 \\ 0 & 0 \end{vmatrix} = 0, \qquad (9)$$

$$(\epsilon + \lambda)\lambda = 0 \longrightarrow \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = -\epsilon \end{matrix}$$

Then the first-order correction of energy will be :

$$E = \frac{1+0}{1-\epsilon} \tag{10}$$

## 3 Problem 6.13

Find the (lowest-order)relativistic correction tothe energy levels of the one-dimensional harmonic oscillator.*Hint*:Use the technique of Problem 2.37.

## 4 Solution

The perturbed Hamiltonian is:

$$H' = \frac{p^4}{8m^3c^2} \tag{11}$$

$\psi_n$ is non-degenerate for the 1-d harmonic oscillator, and the energy eigenvalues are given by :

$$E_n = (n + \frac{1}{2})\hbar\omega \tag{12}$$

Using the ladder operators to construct the momentum operator and implicit theperturbed Hamiltonian we'll obtain ;

$$\begin{aligned}
a_\pm = &\quad \frac{1}{\sqrt{2m}}(p + im\omega x), \qquad (13)\\
(a_+ + a_-) = &\quad \sqrt{\frac{2}{m}}p,\\
p^2 = &\quad \frac{m}{2}(a_+ + a_-)^2 = \frac{m}{2}(a_+^2 + a_-^2 + a_+a_- + a_-a_+),\\
a_+|n> = &\quad \sqrt{n+1}|n+1>,\\
a_-|n> = &\quad \sqrt{n}|n-1>,
\end{aligned}$$

Then we use the definition of $< H' >$ :

$$< H' > = \frac{< n|H'|n >}{< n|n >} \tag{14}$$

From (8) we plug the momentum operator in the expression of Hamiltonian andwe obtain the following :

$$\begin{aligned}
<n|H'|n> = & \frac{1}{8m^3c^2}(<n|p^2)(p^2|n>) \qquad (15) \\
= & frac(\hbar\omega)^2 32mc^2 \left[(n+1)(n+2)+n(n-1)+n^2+(n+1)^2+2n(n+1)\right] \\
= & \frac{(\hbar\omega)^2}{32mc^2}(6n^2+6n+3) \\
<H'> = & \frac{3(\hbar\omega)^2}{32mc^2}(2n^2+2n+1)
\end{aligned}$$

# 5 Problem 6.19

Consider the (eight)$n = 2$ states, $|2ljm_j>$. Find the energy of each state, under weak-field Zeeman splitting, and construct a diagram like Figure 6.11 to show how the energies evolve as $B_{ext}$ increases. Label each line clearly, and indicate its slope.

# 6 Solution

For the $n = 2$ we have the state $|2, l, j, m_j>$ and the energies are given by :

$$E_z' = \mu_B B_{ext} g_j m_j \qquad (16)$$

The corresponding energies with the degenerate levels, Lande' coefficients, and corresponding quantum numbers $(l, j, m_j)$ will be :

$$\begin{aligned}
l=0, j=\frac{1}{2} & \longrightarrow & g_s = 1 + \frac{\frac{1}{2}\frac{3}{2}+\frac{3}{4}-0}{\frac{3}{2}} = 2 \qquad (17) \\
E_z' = & & 2\left(\pm\frac{1}{2}\right)\mu_B B_{ext} \\
l=1, j=\frac{1}{2} & \longrightarrow & g_s = 1 + \frac{\frac{3}{4}+\frac{3}{4}-2}{\frac{3}{2}} = \frac{2}{3} \\
E_z' = & & \frac{2}{3}\left(\pm\frac{1}{2}\right)\mu_B Bext \\
l=1, j=\frac{3}{2} & \longrightarrow & g_s = \frac{\frac{15}{4}+\frac{3}{4}-2}{\frac{15}{2}} = \frac{4}{3}
\end{aligned}$$

$$E_z' = \frac{4}{3}\left(\pm\frac{1}{2}, \pm\frac{3}{2}\right)\mu_B B_{ext}$$

The diagram of weak-field Zeeman splitting looks as follows :

## References

[1] D. J. Griffiths, Introduction To Quantum Mechanics, 1995