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Homework #4#

Course 314,Introduction In Quantum Mechanics,Professor K. Griffioen

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1 Problem 6.2

For the harmonic oscillator $[V(x) = (1/2)kx^2]$, the allowed energies are :

$$E_n = (n + 1/2)\hbar\omega, (n = 0, 1, 2, ...), \tag{1}$$

where $\omega = \sqrt{k}/m$ is the classical frequency. now suppose the spring constant increases slightly : $k \to (1 + \epsilon)k$. (Perhaps we cool the spring, so it becomes less flexible.)

(a) Find the *exact* new energies (trivial, in this case). Expand your formula as a power series in ϵ, up to second order.

(b) Now calculate the first- order perturbation in the energy, using Equation 6.9. What is H'? Compare your result with part (a). *Hint*: It is not necessary - in fact, it is not *permitted* - to calculate a single integral in doing this problem.

2 Solution

(a)

$$\begin{aligned} V &= \frac{1}{2}kx^2 \\ E &= (n + \frac{1}{2})\hbar\omega \\ \omega &= \sqrt{\frac{k}{m}} \\ V' &= \frac{1}{2}(1+\epsilon)kx^2 \end{aligned} \tag{2}$$

then the perturbation of energy will be :

$$\begin{aligned} E' &= (n+\frac{1}{2})\hbar\omega', \\ \omega' &= \sqrt{\frac{(1+\epsilon)k}{m}} = \sqrt{1+\epsilon}\omega \\ E' &= (n+\frac{1}{2})(\hbar\omega)(1+\frac{\epsilon}{2}-\frac{\epsilon^2}{8}+...) \end{aligned} \tag{3}$$

and $\sqrt{1+\epsilon} = 1+\frac{\epsilon}{2}-\frac{\epsilon}{8}+...$, represent the Taylor expansion of $(1+\epsilon)^{1/2}$.

(b) The first-order correction of energy is given by :

$$E_1 =< \psi_n|H'|\psi_n >=< \psi_n|\frac{1}{2}k\epsilon x^2|\psi_n >= \epsilon < \psi_n|\frac{1}{2}kx^2|\psi_n >= \epsilon < V > \tag{4}$$

The virial theorem states that :

$$\begin{aligned} 2 < T >= &< \bar{r}.\bar{\nabla} V >=< x.\frac{d}{dx}V >=< x.\frac{2}{2}kx >= 2 < V >, but \\ E = &< V > + < T >, then \\ < V >= &\frac{1}{2}(n+\frac{1}{2})\hbar\omega \\ E_1 = &\frac{\epsilon}{2}(n+\frac{1}{2})\hbar\omega \end{aligned} \tag{5}$$

This result is just what we get for the ϵ power of the exact expansion.

3 Problem 6.3

Two identical bosons are placed in an infinite square well(Equation 2.15.) They interact weakly with one another, via the potential

$$V(x_1, x_2) = -aV_0\delta(x_1 - x_2) \tag{6}$$

(where V_0 is a constant with dimensions of energy and a is the width of the well).

(a) First, ignoring the interaction between particles, find the ground state and first excited state - both the wave functions and the associated energies.

(b) Use first - order perturbation theory to calculate the affect of the particle - particle interaction on the ground and first excited state energies.

4 Solution

(a)

$$\begin{aligned} V(x_1, x_2) = & \quad -aV_0\delta(x_1 - x_2), \\ \psi_n(x) = & \quad \sqrt{\frac{2}{a}} sin\frac{n\pi x}{a} \\ E_n = & \quad \frac{n^2\pi^2\hbar^2}{2ma^2} \end{aligned} \tag{7}$$

where ψ_n represent the general wave function which characterize the bosons in an infinite square well potential.

The ground state will have the following wave function :

$$\psi_{gs} = \psi_1(x_1)\psi_2(x_2) = \frac{2}{a} sin\frac{\pi x_1}{a} sin\frac{\pi x_2}{a} \tag{8}$$

The associated ground state energy will be :

$$E_{gs} = \frac{\pi^2\hbar^2}{ma^2} \tag{9}$$

For the first excited state the wave function will be :

$$\psi_{1st} = \frac{1}{\sqrt{2}}\Big(\psi_1(x_1)\psi_2(x_2) + \psi_1(x_2)\psi_2(x_1)\Big) = \frac{\sqrt{2}}{a}\Big(sin\frac{\pi x_1}{a} sin\frac{2\pi x_2}{a} + sin\frac{2\pi x_1}{a} sin\frac{\pi x_2}{a}\Big) \tag{10}$$

The corresponding energy for the first excited state will be :

$$E_{1st} = \frac{5\pi^2\hbar^2}{2ma^2} \tag{11}$$

(b) Under the interaction between particles the first-order correction of the energy in the ground state will be :

$$
\begin{aligned}
E'_{gs} = & \int_0^a dx_1 \int_0^a \psi_1(x_1)^2\psi_1(x_2)^2(-aV_0)\delta(x_1-x_2)dx_2 \\
= & -aV_0 \int_0^a \psi_1(x_1)^4 dx_1 \\
= & -aV_0 \left(\frac{2}{a}\right)^2 \int_0^a sin\frac{\pi x}{a}^4 dx \\
= & -\frac{3}{2}V_0
\end{aligned} \tag{12}
$$

and for the first excited state :

$$
\begin{aligned}
E_{1st} = & \left(\frac{2}{a}\right)^2 \int_0^a dx_1 \int_0^a (-aV_0)\delta(x_1-x_2)\left(\frac{1}{\sqrt{2}}\psi_1(x_1)\psi_2(x_2) + \frac{1}{\sqrt{2}}\psi_1(x_2)\psi_2(x_1)\right)^2 dx_2 \\
= & \left(\frac{2}{a}\right)^2 \int_0^a dx_1 \int_0^a (-aV_0)\left(\frac{2}{sqrt2}\psi_1(x_1)\psi_2(x_1)\right)^2 dx_1 \\
= & -\frac{8}{a}V_0 \int_0^a sin\frac{\pi x}{a}^2 sin\frac{2\pi x}{a} dx \\
E_{1st} = & -2V_0
\end{aligned} \tag{13}
$$

5 Problem 6.7

Consider a particle of mass m that is free to move in a one - dimensional region of length L that closes onitself (for instance, a bead which slides frictionlessly on a circular wire of circumference L ; Problem 2.43).

(a) Show that the stationary states can be written in the form

$$
\psi_n(x) = \frac{1}{\sqrt{L}}e^{2\pi inxL}, (-L/2 < x < L/2), \tag{14}
$$

where $n = 0, \pm 1, \pm 2, ...$, and the allowed energies are

$$
E_n = \frac{2}{m}\left(\frac{n\pi\hbar}{L}\right)^2. \tag{15}
$$

Notice that - with the exception of the ground state ($n = 0$) - these are all doubly degenerate.

Xcy **(b)** Now suppose we introduce the perturbation

$$H' = -V_0 e^{-x^2/a^2}, \tag{16}$$

where $a << L$. (This puts a little " dimple " in the potential at $x = 0$, as though we bent the wire slightly to make a " trap. ") Find the first-order correction to E_n, using Equation 6.26. *Hint*: To evaluate the integrals, exploit the fact that $a << L$ to extend the limits from $\pm L/2$ to $\pm\infty$; after all, H' is essentially zero outside $-a << x << a$.

(c) What are the " good " linear combinations of ψ_n and ψ_{-n} for this problem ? Show that with these states you get the first-order correction using Equation 6.9.

(d) Find a Hermitin operator A that fits the requirement of the theorem, and show that the simultaneous eigenstates of H^0 and A are precisely the ones you found in (c).

6 Solution

(a) From Shrodinger equation we have :

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} &= E\psi, \\ k^2 &= \frac{2mE}{\hbar^2}, \\ \frac{d^2\psi}{dx^2} + k^2\psi &= 0, \\ \psi &= e^{\pm ikx} \end{aligned} \tag{17}$$

Applying the boundary condition $\psi(x+L) = \psi(x)$, we'll find :

$$\begin{aligned} e^{\pm ik(x+L)} &= e^{\pm ikx}, \\ e^{\pm ikx}e^{\pm ikL} &= e^{\pm ikx}, \\ e^{\pm ikL} &= 1, \\ kL &= 2\pi n, n = 0, \pm 1, \pm 2, ... \end{aligned} \tag{18}$$

Applying the normalization condition of wave function we'll have :

$$\int_{-L/2}^{L/2} \psi^* \psi dx = 1, \tag{19}$$
$$\int_{-L/2}^{L/2} A^2 dx = 1,$$
$$A = \frac{1}{\sqrt{L}}$$

Then we'll plug into $\psi = e^{\pm ikx}$ and we'll obtain :

$$\psi_n = \frac{1}{\sqrt{L}} e^{2\pi inx/L}, n = 0, \pm 1, \pm 2, ... \tag{20}$$
$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 (2\pi) n^2}{2mL^2} = \frac{2}{m}\left(\frac{n\hbar\pi}{L}\right)^2$$

(b) Under the small perturbation $H' = -V_0 e^{-x^2/a^2}$, where $a << L$, the first-order correction of E_n is given by :

$$E'_{\pm} = \frac{1}{2}\left[W_{aa} + W_{bb} \pm \sqrt{(W_{aa} - W_{bb})^2 + 4|W_{ab}|^2}\right], \tag{21}$$

where the matrix elements $W_{aa}, W_{bb}, W_{ab} = W_{ba}$ can be computed as follows :

$$W_{aa} = W_{bb} = < \psi_a | H' | \psi_a >= \int_{-L/2}^{L/2} \psi_a^* H' \psi_a dx = -\frac{V_0}{L} \int_{-L/2}^{L/2} e^{-x^2/a^2} dx, \tag{22}$$
$$W_{ab} = W_{ba} = < \psi_a | H' | \psi_b >=< \psi_b | H' || psi_a >= -\frac{V_0}{L} \int_{L/2}^{L/2} \psi_a^* H' \psi_b dx = -\frac{V_0}{L} \int_{-L/2}^{L/2} e^{-4\pi inx/L} e^{-x^2/a^2} dx$$
$$\psi_a = \frac{1}{\sqrt{L}} e^{+2\pi inx/L}$$
$$\psi_b = \frac{1}{sqrtL} e^{-2\pi inx/L}$$

We can make the approximation $L/2 \rightarrow \infty$, since $L >> a$, so vthen we'll have :

$$W_{aa} = W_{bb} = -\frac{aV_0\sqrt{\pi}}{L}, \tag{23}$$

$$W_{ab} = W_{ba} = \quad -\frac{aV_0\sqrt{\pi}}{L}e^{-4\pi^2n^2a^2/L^2}$$

We'll make the subscript $e^{-4\pi^2n^2a^2/L^2} = \epsilon$, and then the first - order correction of E_n will be :

$$E'_{\pm} = -\frac{aV_0}{L}(1 \mp \epsilon) \tag{24}$$

(c) We resolve the eigenvalue problem knowing that the transformation matrix has the following form :

$$W = \begin{pmatrix} W_{aa} & W_{ab} \\ W_{ba} & W_{bb} \end{pmatrix} = -\frac{aV_0\pi}{l}\begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} \tag{25}$$

Then we'll determine the eigenstates of the matrix operator W :

$$\begin{aligned} \begin{pmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \quad 0 \longrightarrow \frac{1}{\sqrt{L}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \begin{pmatrix} -\epsilon & \epsilon \\ \epsilon & -\epsilon \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \quad 0 \longrightarrow \frac{1}{\sqrt{L}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned} \tag{26}$$

The "good" combinations of ψ_n and ψ_{-n} will be :

$$\begin{aligned} \psi_n &= \frac{1}{\sqrt{2L}}\left(e^{2\pi inx/L} + e^{-2\pi inx/L}\right) \\ \psi_{-n} &= \frac{1}{\sqrt{2L}}\left(e^{2\pi inx/L} - e^{-2\pi inx/L}\right) \end{aligned} \tag{27}$$

(d) We have to find an operator A that fits the requirement to be Hermitian, and the H^0 and A to have the same eigenstates :

$$\begin{aligned} A^{\dagger} &= \quad A, \\ \left[A, H'\right] &= \quad 0 \end{aligned} \tag{28}$$

Under theorem conditions :

$$A\psi_n = \quad 1\psi_n, \tag{29}$$

$$A\psi_{-n} = -1\psi_{-n}$$

So the operator A takes x to the - x (parity operator), and since $e^{-(-x)^2/a^2} = e^{-x^2/a^2}$, the eigenfunctions $\psi_{\pm n}$ are simultaneous eigenstates for A and H^0.

References

[1] D. J. Griffiths, Introduction To Quantum Mechanics, 1995