1 Problem 6.2

For the harmonic oscillator \( V(x) = (1/2)kx^2 \), the allowed energies are:

\[
E_n = (n + 1/2)\hbar\omega, \quad (n = 0, 1, 2, ...),
\]

(1)

where \( \omega = \sqrt{k/m} \) is the classical frequency. Now suppose the spring constant increases slightly:

\[ k \rightarrow (1 + \epsilon)k. \]

(Perhaps we cool the spring, so it becomes less flexible.)

(a) Find the exact new energies (trivial, in this case). Expand your formula as a power series in \( \epsilon \), up to second order.

(b) Now calculate the first-order perturbation in the energy, using Equation 6.9. What is \( H' \)? Compare your result with part (a). \textit{Hint}: It is not necessary - in fact, it is not \textit{permitted} - to calculate a single integral in doing this problem.

2 Solution

(a)

\[
V = \frac{1}{2}kx^2
\]

\[
E = (n + 1/2)\hbar\omega
\]

\[
\omega = \sqrt{\frac{k}{m}}
\]

\[
V' = \frac{1}{2}(1 + \epsilon)kx^2
\]
then the perturbation of energy will be:

\[ E' = (n + \frac{1}{2})\hbar\omega', \]
\[ \omega' = (1 + \epsilon)\frac{k}{m} = \sqrt{1 + \epsilon} \omega \]
\[ E' = (n + \frac{1}{2})\hbar\omega(1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \ldots) \]

and \( \sqrt{1 + \epsilon} = 1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \ldots \), represent the Taylor expansion of \((1 + \epsilon)^{1/2}\).

(b) The first-order correction of energy is given by:

\[ E_1 = \langle \psi_n | H' | \psi_n \rangle = \langle \psi_n | \frac{1}{2} k\epsilon x^2 | \psi_n \rangle = \epsilon \langle \psi_n | \frac{1}{2} k\epsilon x^2 | \psi_n \rangle = \epsilon < V > \]

The virial theorem states that:

\[ 2 < T > = \langle \psi | \nabla \psi | x \nabla \psi > = \langle x \frac{d}{dx} V > = \langle x \frac{2}{2} k\epsilon x^2 > = 2 < V >, \text{ but} \]
\[ E = < V > + < T >, \text{ then} \]
\[ < V > = \frac{1}{2} (n + \frac{1}{2})\hbar\omega \]
\[ E_1 = \frac{\epsilon}{2} (n + \frac{1}{2})\hbar\omega \]

This result is just what we get for the \( \epsilon \) power of the exact expansion.

3 Problem 6.3

Two identical bosons are placed in an infinite square well (Equation 2.15.) They interact weakly with one another, via the potential

\[ V(x_1, x_2) = -a V_0 \delta(x_1 - x_2) \]

(where \( V_0 \) is a constant with dimensions of energy and \( a \) is the width of the well).

(a) First, ignoring the interaction between particles, find the ground state and first excited state - both
the wave functions and the associated energies.

(b) Use first-order perturbation theory to calculate the effect of the particle-particle interaction on the ground and first excited state energies.

4 Solution

(a) 

\[
V(x_1, x_2) = -aV_0 \delta(x_1 - x_2),
\]

\[
\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a},
\]

\[
E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}
\]

where \(\psi_n\) represent the general wave function which characterize the bosons in an infinite square well potential.

The ground state will have the following wave function:

\[
\psi_{gs} = \psi_1(x_1)\psi_2(x_2) = \frac{2}{a} \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{a}
\]

The associated ground state energy will be:

\[
E_{gs} = \frac{\pi^2 \hbar^2}{ma^2}
\]

For the first excited state the wave function will be:

\[
\psi_{1st} = \frac{1}{\sqrt{2}} \left( \psi_1(x_1)\psi_2(x_2) + \psi_1(x_2)\psi_2(x_1) \right) = \frac{\sqrt{2}}{a} \left( \sin \frac{\pi x_1}{a} \sin \frac{2\pi x_2}{a} + \sin \frac{2\pi x_1}{a} \sin \frac{\pi x_2}{a} \right)
\]

The corresponding energy for the first excited state will be:

\[
E_{1st} = \frac{7\pi^2 \hbar^2}{2ma^2}
\]

(b) Under the interaction between particles the first-order correction of the energy in the ground state will be:
\[ E'_{gs} = \int_0^a dx_1 \int_0^a \psi_1(x_1)^2 \psi_1(x_2)^2 (-aV_0)\delta(x_1 - x_2)dx_2 \]
\[ = -aV_0 \int_0^a \psi_1(x_1)^4 dx_1 \]
\[ = -aV_0 \left( \frac{2}{a} \right)^2 \int_0^a \sin^2 \frac{\pi x}{a} dx \]
\[ = -\frac{3}{2}V_0 \]

and for the first excited state:

\[ E_{1st} = \left( \frac{2}{a} \right)^2 \int_0^a dx_1 \int_0^a (-aV_0)\delta(x_1 - x_2) \left( \frac{1}{\sqrt{2}} \psi_1(x_1)\psi_2(x_2) + \frac{1}{\sqrt{2}} \psi_1(x_2)\psi_2(x_1) \right)^2 dx_2 \]
\[ = \left( \frac{2}{a} \right)^2 \int_0^a dx_1 \int_0^a (-aV_0) \left( \frac{2}{\sqrt{2}} \psi_1(x_1)\psi_2(x_1) \right)^2 dx_1 \]
\[ = -\frac{8}{a}V_0 \int_0^a \sin^2 \frac{\pi x}{a} \sin^2 \frac{2\pi x}{a} dx \]
\[ E_{1st} = -2V_0 \]

5 Problem 6.7

Consider a particle of mass \( m \) that is free to move in a one-dimensional region of length \( L \) that closes on itself (for instance, a bead which slides frictionlessly on a circular wire of circumference \( L \); Problem 2.43).

(a) Show that the stationary states can be written in the form

\[ \psi_n(x) = \frac{1}{\sqrt{L}} e^{2\pi inxL}, \quad (-L/2 < x < L/2), \]

where \( n = 0, \pm 1, \pm 2, \ldots \), and the allowed energies are

\[ E_n = \frac{2}{m} \left( \frac{n\pi \hbar}{L} \right)^2. \]

Notice that - with the exception of the ground state \( n = 0 \) - these are all doubly degenerate.

Xcy (b) Now suppose we introduce the perturbation
\[
H' = -V_0 e^{-x^2/a^2},
\]
where \( a << L \). (This puts a little "dimple" in the potential at \( x = 0 \), as though we bent the wire slightly to make a "trap"). Find the first-order correction to \( E_n \), using Equation 6.26. Hint: To evaluate the integrals, exploit the fact that \( a << L \) to extend the limits from \( \pm L/2 \) to \( \pm \infty \); after all, \( H' \) is essentially zero outside \(-a << x << a\). 

(c) What are the "good" linear combinations of \( \psi_n \) and \( \psi_{-n} \) for this problem? Show that with these states you get the first-order correction using Equation 6.9.

(d) Find a Hermitian operator \( A \) that fits the requirement of the theorem, and show that the simultaneous eigenstates of \( H^0 \) and \( A \) are precisely the ones you found in (c).

6 Solution

(a) From Schrödinger equation we have:

\[
\frac{-\hbar^2}{2m} \frac{d^2 \psi}{dx^2} = E \psi, \tag{17}
\]

\[
k^2 = \frac{2mE}{\hbar^2},
\]

\[
\frac{d^2 \psi}{dx^2} + k^2 \psi = 0, \quad \psi = e^{\pm ikx}
\]

Applying the boundary condition \( \psi(x + L) = \psi(x) \), we’ll find:

\[
e^{\pm ik(x+L)} = e^{\pm ikx}, \tag{18}
\]

\[
e^{\pm ikx} e^{\pm iKL} = e^{\pm ikx},
\]

\[
e^{\pm iKL} = 1,
\]

\[
kL = 2\pi n, n = 0, \pm 1, \pm 2, ...
\]
Applying the normalization condition of wave function we’ll have:

\[
\int_{-L/2}^{L/2} \psi^* \psi \, dx = 1, \\
\int_{-L/2}^{L/2} A^2 \, dx = 1, \\
A = \frac{1}{\sqrt{L}}
\]  

Then we’ll plug in \( \psi = e^{\pm ikx} \) and we’ll obtain:

\[
\psi_n = \frac{1}{\sqrt{L}} e^{\pm 2\pi inx/L}, \quad n = 0, \pm 1, \pm 2, \ldots
\]

\[
E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 (2\pi n)^2}{2mL^2} = \frac{2}{m} \left( \frac{n\pi}{L} \right)^2
\]

(b) Under the small perturbation \( H' = -V_0 e^{-x^2/a^2}, \) where \( a << L, \) the first-order correction of \( E_n \) is given by:

\[
E_{n,\pm} = \frac{1}{2} \left[ W_{aa} + W_{bb} \pm \sqrt{(W_{aa} - W_{bb})^2 + 4|W_{ab}|^2} \right],
\]

where the matrix elements \( W_{aa}, W_{bb}, W_{ab} = W_{ba} \) can be computed as follows:

\[
W_{aa} = W_{bb} = \langle \psi_a | H' | \psi_a \rangle = \int_{-L/2}^{L/2} \psi_a^* H' \psi_a \, dx = -\frac{V_0}{L} \int_{-L/2}^{L/2} e^{-x^2/a^2} \, dx,
\]

\[
W_{ab} = W_{ba} = \langle \psi_a | H' | \psi_b \rangle = \langle \psi_b | H' | \psi_a \rangle = -\frac{V_0}{L} \int_{-L/2}^{L/2} \psi_a^* H' \psi_b \, dx = -\frac{V_0}{L} \int_{-L/2}^{L/2} e^{-4\pi inx/L} e^{-x^2/a^2} \, dx
\]

\[
\psi_a = \frac{1}{\sqrt{L}} e^{+2\pi inx/L}, \\
\psi_b = \frac{1}{\sqrt{\sqrt{L}}} e^{-2\pi inx/L}
\]

We can make the approximation \( L/2 \to \infty, \) since \( L \gg a, \) so when we’ll have:

\[
W_{aa} = W_{bb} = -\frac{a V_0 \sqrt{\pi}}{L},
\]
\[ W_{ab} = W_{ba} = -\frac{aV_0\sqrt{\pi}}{L} e^{-4\pi^2 n^2 a^2/L^2} \]

We’ll make the subscript \( e^{-4\pi^2 n^2 a^2/L^2} = \epsilon \), and then the first - order correction of \( E_n \) will be :

\[ E_\pm = -\frac{aV_0}{L} (1 \mp \epsilon) \]  

(24)

(c) We resolve the eigenvalue problem knowing that the transformation matrix has the following form :

\[
W = \begin{pmatrix}
W_{aa} & W_{ab} \\
W_{ba} & W_{bb}
\end{pmatrix} = -\frac{aV_0\pi}{L} \begin{pmatrix}
1 & \epsilon \\
\epsilon & 1
\end{pmatrix}
\]  

(25)

Then we’ll determine the eigenstates of the matrix operator \( W \) :

\[
\begin{pmatrix}
1 - \epsilon & \epsilon \\
\epsilon & 1 - \epsilon
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} = 0 \rightarrow \frac{1}{\sqrt{L}} \begin{pmatrix}
1 \\
-1
\end{pmatrix}
\]  

(26)

\[
\begin{pmatrix}
-\epsilon & \epsilon \\
\epsilon & -\epsilon
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} = 0 \rightarrow \frac{1}{\sqrt{L}} \begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

The "good" combinations of \( \psi_n \) and \( \psi_{-n} \) will be :

\[
\psi_n = \frac{1}{\sqrt{2L}} \left( e^{2\pi i n x/L} + e^{-2\pi i n x/L} \right)
\]  

(27)

\[
\psi_{-n} = \frac{1}{\sqrt{2L}} \left( e^{2\pi i n x/L} - e^{-2\pi i n x/L} \right)
\]

(d) We have to find an operator \( A \) that fits the requirement to be Hermitian, and the \( H^0 \) and \( A \) to have the same eigenstates :

\[
A^\dagger = A,
\]  

(28)

\[
\left[ A, H^0 \right] = 0
\]

Under theorem conditions :

\[
A\psi_n = 1\psi_n,
\]  

(29)
\[ A\psi_{-n} = -1\psi_{-n} \]

So the operator \( A \) takes \( x \) to the \(-x\) (parity operator), and since \( e^{-(-x)^2/a^2} = e^{-x^2/a^2} \), the eigenfunctions \( \psi_{\pm n} \) are simultaneous eigenstates for \( A \) and \( H^0 \).

References