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Problem Set #3

Physics 314: Introduction to Quantum Mechanics
Solutions by Cornel Butuceanu
Department of Physics, College of William and Mary, Williamsburg, Virginia 23187

1 Problem 5.22

Obtain Equation 5.75 by induction. The combinatorial question is this: How many different ways can you put N identical balls into d baskets (never mind the subscript n for this problem). You could stick all N of them into the third basket, or all but one in the second basket and one in the fifth, or two in the first and three in the third and all the rest in the seventh, etc. Work it out explicitly for the cases $N = 1, N = 2, N = 3, andN = 4$;by the stage you should be able to deduce the general formula.

1.1 Solution

$$\tag{1}$$

$$\mathbf{P_n} = \frac{\mathbf{1}}{\mathbf{N_n}!}\frac{(\mathbf{N_n} + \mathbf{d_n} - \mathbf{1})}{(\mathbf{d_n} - \mathbf{1})} \tag{2}$$

Where P_n represent the number of distinct ways of assigning the N_n particles to the d_n one-particle states in the n th bin.

2 Problem 5.23

Use the method of Lagrange multipliers to find the area of the largest rectangle, with sides parallel to the axes, that can be inscribed in the ellipse $(x/a)^2 + (y/b)^2 = 1$.

2.1 Solution

The function to be maximized has the next form:

$$F(x, y) = 4xy, \tag{3}$$

under following constraint:

$$g(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4}$$

and using Lagrange multipliers method we'll have:

$$\begin{aligned} G(x,y) &= F(x,y) + \lambda g(x,y), \\ \frac{\partial G}{\partial x} &= 0 \\ \frac{\partial G}{\partial y} &= 0 \\ \frac{\partial G}{\partial \lambda} &= 0 \end{aligned} \quad (5)$$

λ represent the Lagrange multiplier. After computing on components we'll have:

$$\begin{cases} y + \lambda(\frac{2x}{a^2}) = 0 \\ x + \lambda(\frac{2y}{b^2}) = 0 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \end{cases} \quad (6)$$

we'll determine three unknowns λ, x, y:

$$\lambda = 2ab, x = \frac{a}{\sqrt{2}}, y = \frac{b}{\sqrt{2}} \quad (7)$$

The area of the largest rectangle, that can be inscribed in that ellipse will be equal with numerical value of $F(x,y) = xy$, at the point described of coordinates $(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$.

$$F_{max} = \frac{ab}{2} \quad (8)$$

3 Problem 5.24

(a) Find the percent error in Stirling's approximation for $z = 10$.
(b) What is the smallest integer z such that the error is less than 1 percent.

3.1 Solution

In Stirling's approximation:

$$\ln(z!) \approx z\ln(z) - z, \text{ for } z \gg 1 \quad (9)$$

The percent error in Stirling's approximation for $z = 10$ is 13.76, and the smallest integer z for which the error is less than 1 percent is 90.

Table 1: Comparison of Stirling's formula with the exact result

Z	**Error(percent)**
10	13.76
15	8.16
20	5.71
25	4.36
30	3.51
35	2.92
40	2.50
45	2.18
50	1.93
55	1.73
60	1.57
65	1.43
70	1.32
75	1.22
80	1.13
85	1.06
90	0.99

4 Problem 5.28

Derive the **Stefan-Boltzmann formula** for the total energy density in blackbody radiation:

$$\frac{E}{V} = (\frac{\pi^2 k_B^4}{15\hbar^3 c^3})T^4 = (7.57 \times 10^{16} \mathrm{Jm}^{-3}\mathrm{K}^{-4})T^4. \tag{10}$$

4.1 Solution

The energy density $\rho(\omega)$ is defined by:

$$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3 (e^{\hbar\omega/k_b T} - 1)} \tag{11}$$

The total energy in blackbody radiation becomes:

$$\begin{aligned} \frac{E}{V} &= \int_0^\infty \rho(\omega) d\omega = \int_0^\infty \frac{\hbar\omega^3 d\omega}{\pi^2 c^3 (e^{\hbar\omega/k_b T} - 1)}, \\ \frac{E}{V} &= \frac{\hbar}{\pi^2 c^3} \int_0^\infty \omega^3 (e^{\hbar\omega/k_B T} - 1)^{-1} d\omega \end{aligned} \tag{12}$$

Let's make the substitutions:

$$\begin{aligned} x &= \frac{\hbar\omega}{k_b T}, \\ \omega^3 &= \frac{x k_b T}{\hbar} \end{aligned} \tag{13}$$

after differentiation we'll obtain:

$$
\begin{aligned}
dx = & \quad \frac{\hbar d\omega}{k_b T}, \\
d\omega = & \quad (\frac{k_b T}{\hbar}) dx
\end{aligned}
\tag{14}
$$

we plug into(9), and the integral transforms as follows:

$$
\begin{aligned}
\frac{E}{V} = & \quad \frac{k_B^4 T^4}{\pi^2 c^3 \hbar^3} \int_0^\infty \frac{x^3}{(e^x - 1)} dx, \\
\frac{E}{V} = & \quad \frac{k_B^4 T^4}{\pi^2 c^3 \hbar^3}, (4)\zeta(4), \\
\frac{E}{V} = & \quad \frac{k_B^4 T^4}{\pi^2 c^3 \hbar^3} (\frac{\pi^4}{15}) = (\frac{\pi^2 k_B^4}{15 c^3 \hbar^3}) T^4
\end{aligned}
\tag{15}
$$

where , is Euler's **gamma function** and ζ is the **Riemann zeta function** , and also we used the result from Problem 5.26:

$$
\int_0^\infty \frac{x^{s-1}}{(e^x - 1)} dx = , (s)\zeta(s), \tag{16}
$$

For our case $s = 4$ and $\int_0^\infty \frac{x^3}{(e^x-1)} = \frac{\pi^4}{15}$.

References

[1] D. J. Griffiths, Introduction to Quantum Mechanics, 1995