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Problem Set #2

Physics 314: Introduction to Quantum Mechanics
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1 Problem 5.19

(a) construct the completely antisymmetric wave function $\psi(x_a, x_b, x_c)$ for three identical fermions, one in the state ψ_5, one in the state ψ_7, and one in the state ψ_{13}.
(b) Construct the completely symmetric wave function $\psi(x_a, x_b, x_c)$ for three identical bosons,(i) if all three are in state ψ_9, (ii) if two are in state ψ_7 and one in ψ_{13}, and (iii) if one is in state ψ_5, one in state ψ_7, and one in state ψ_{13}.

1.1 Solution

(a)
For three identical fermions, the completely antisymmetric wave function will be:

$$\psi(\mathbf{x_a}, \mathbf{x_b}, \mathbf{x_c}) = A \det \begin{pmatrix} \psi_5(x_a)\psi_7(x_a)\psi_{13}(x_a) \\ \psi_5(x_b)\psi_7(x_b)\psi_{13}(x_b) \\ \psi_5(x_c)\psi_7(x_c)\psi_{13}(x_c) \end{pmatrix} \tag{1}$$

with other words the wave function can be written as follows:

$$\begin{aligned} \psi_(x_a, x_b, x_c) = \; & A[\psi_5(x_a)\psi_7(x_b)\psi_{13}(x_c) + \psi_5(x_b)\psi_7(x_c)\psi_{13}(x_a) + \\ & \psi_5(x_c)\psi_7(x_a)\psi_{13}(x_b) - \psi_5(x_c)\psi_7(x_b)\psi_{13}(x_a) - \\ & \psi_5(x_b)\psi_7(x_a)\psi_{13}(x_c) - \psi_5(x_a)\psi_7(x_c)\psi_{13}(x_b)]. \end{aligned} \tag{2}$$

where A is the normalization constant.
(b) For three identical bosons, the completely symmetrical wave function can be written, respectively as follows:
(i)

$$\psi(x_a, x_b, x_c) = A\psi_9(x_a)\psi_9(x_b)\psi_9(x_c) \tag{3}$$

(ii)

$$\begin{aligned} \psi(x_a, x_b, x_c) = \; & A[\psi_3(x_a)\psi_3(x_b)\psi_{15}(x_c) + \\ & \psi_3(x_b)\psi_3(x_c)\psi_{15}(x_a) + \\ & \psi_3(x_c)\psi_3(x_a)\psi_{15}(x_b)] \end{aligned} \tag{4}$$

(iii)

$$\psi(x_a,x_b,x_c) = A[\psi_5(x_a)\psi_7(x_b)\psi_{13}(x_c) + \psi_5(x_b)\psi_7(x_c)\psi_{13}(x_a) + \psi_5(x_c)\psi_7(x_a)\psi_{13}(x_b) + \psi_5(x_c)\psi_7(x_b)\psi_{13}(x_a) + \psi_5(x_b)\psi_7(x_a)\psi_{13}(x_c) + \psi_5(x_a)\psi_7(x_c)\psi_{13}(x_b)] \quad (5)$$

2 Problem 5.20

Suppose you had three particles in one-dimensional harmonic oscillator potential, in thermal equilibrium, with a total energy $E = (9/2)\hbar\omega$.
(a) If they are distinguishable particles (but all the same mass), what are the possible occupation-number configurations, and how many distinct (three particles) states are there for each one? What is the most probable configuration? If you picked a particle at a random and measured its energy, what values might you get, and what is the probability of each one? What is the most probable energy?
(b) Do the same for the case of identical fermions (ignoring spin, as in Example in Section 5.14).
(c) Do the same for the case of identical bosons (ignoring spin)

2.1 Solution

For a harmonic oscillator, the energy is :

$$\begin{aligned}
E_n &= (n+1/2)\hbar\omega \\
E_1 &= (n_1+1/2)\hbar\omega \\
E_2 &= (n_2+1/2)\hbar\omega \\
E_3 &= (n_3+1/2)\hbar\omega \\
E_{tot} &= E_1+E_2+E_3 = (n_1+n_2+n_3+3/2)\hbar\omega \\
E_{tot} &= 9/2\hbar\omega \\
n_1+n_2+n_3 &= 3 \\
n_i &= 0;1;2;3 \\
i &= 1-3
\end{aligned} \quad (6)$$

(a)
For three distinguishable particles there are 3 possible occupation number configuration :
for configuration $N_1 = 3$ we have 1 distinct state, for configuration $N_0 = 1, N_1 = 1, N_2 = 1$ we have 6 distinct states, and for configuration $N_0 = 2, N_3 = 1$ we have 3 distinct states.
The second configuration is the most probable because has the biggest number of states possible.
The possible energies of a particle chosen at random and their respective probabilities are as follows :

Energy	$\longrightarrow$	Probability
$1/2\hbar\omega$		$4/10$
$3/2\hbar\omega$		$3/10$
$5/2\hbar\omega$		$2/10$
$7/2\hbar\omega$		$1/10$

The most probable energy is $E_0 = (1/2)\hbar\omega$.

(b)

For three identical fermions there is only one possible occupation number configuration :

Configuration	Number of Distinct States
$N_0 = 1, N_1 = 1, N_2 = 1$	1

The most probable configuration, of course, can be one shown above, and the possible energies of a particle, in this configuration, at certain random, and their probabilities, are as follows :

Energy	$\longrightarrow$	Probability
$1/2\hbar\omega$		$1/3$
$3/2\hbar\omega$		$1/3$
$5/2\hbar\omega$		$1/3$

All three of these probable energies are equally likely.

(c)

For three identical bosons there are 3 possible occupation number configurations :

Configuration	Number of Distinct States
$N_1 = 3$	1
$N_0 = 1, N_1 = 1, N_2 = 1$	1
$N_0 = 2, N_3 = 1$	1

All three of these probable configurations are equally likely, and the possible energies of a particle, at certain random, and their probabilities are as follows :

Energy	Probability
$1/2\hbar\omega$	$3/9$
$3/2\hbar\omega$	$4/9$
$5/2\hbar\omega$	$1/9$
$7/2\hbar\omega$	$1/9$

The most probable energy is $E_1 = 3/2\hbar\omega$.

References

[1] D. J. Griffiths, Introduction To Quantum Mechanics, 1995