February 29, 2000

Problem Set #1

Physics 314: Introduction to Quantum Mechanics
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1 Problem 5.13

The density of copper is 8.96 gm/cm^3, and its atomic weight is 63.5 gm/mole.

(a) Calculate the Fermi energy for copper (equation 5.43). Assume $q = 1$, and give your answer in electron volts.

(b) What is the corresponding electron velocity [set $E_F = (1/3)mv^2$]? Is it safe to assume that the electrons in copper are nonrelativistic?

(c) At what temperature would the characteristic thermal energy ($k_B T$, wherek_B is the Boltzmann constant and T is the Kelvin temperature) equal the Fermi energy, for copper? *Note:* This is called the **Fermi temperature**. As long as the *actual* temperature is substantially bellow the Fermi temperature, the material can be regarded as "cold," with most of the electrons in the ground-state configuration. Since the melting point of copper is $1356K$, solid copper is *always* cold.

(d) Calculate the degeneracy pressure (Equation 5.46) of copper, in the gas model.

1.1 Solution

Given:

$$E_F = \frac{\hbar^2}{2m_e}(3\rho\pi^2)^{2/3} \tag{1}$$

where

$$\begin{aligned} \rho_{Cu} &= 8.96\frac{\text{g}}{\text{cm}^3} \\ A_{Cu} &= 63.5\frac{\text{g}}{\text{mole}} \\ N_A &= 6.022 \times 10^{23}\text{mole}^{-1} \\ h &= 6.626 \times 10^{-34}\text{J} \cdot \text{s} \\ m_e &= 9.109 \times 10^{-31}\text{kg} \\ k_B &= 1.38 \times 10^{-23}\text{J/K} \\ q &= 1. \end{aligned} \tag{2}$$

(a)
We also know that:

$$\rho = \frac{Nq}{V} = \frac{1}{V_{Cu}} \tag{3}$$

$$
\begin{aligned}
m_{Cu} = &\quad \rho_{Cu} V_{Cu} = \frac{A}{N_A} \\
V_{Cu} = &\quad \frac{A}{N_A \rho_{Cu}} \\
\rho = &\quad \frac{N_A \rho_{Cu}}{A}
\end{aligned}
$$

and substituting this into Eq. 1 we find that

$$
E_F = \frac{h^2}{8\pi^2 m_e} \left(3\pi^2 \frac{N_A \rho_{Cu}}{A} \right)^{2/3} \tag{4}
$$

Numerical result: $E_F = 1.13022 \times 10^{-18}\text{J} = 7.05$ eV

(b)

$$
\begin{aligned}
E_F = &\quad \frac{mv^2}{2} \\
v_e = &\quad \sqrt{\frac{2E_F}{m_e}}
\end{aligned} \tag{5}
$$

Numerical result: $v_e = 1.575 \times 10^6$ m/s , but $c = 3 \times 10^8$m/s, so we'll then have $v_e/c = 5.25 \times 10^{-3}$.
Conclusion: We can assume that the electrons in copper are nonrelativistic.

(c)

$$
\begin{aligned}
E_T = &\quad k_B T \\
E_F = &\quad E_T \\
T_F = &\quad \frac{E_F}{k_B}
\end{aligned} \tag{6}
$$

Numerical solution: $T_F = 8.18 \times 10^4 K$, $T_F >> T_M$ (melting temperature).

(d)

$$
\begin{aligned}
P = &\quad \frac{(3\pi^2)^{2/3} h^2}{20 m_e \pi^2} \rho^{5/3} \\
P = &\quad (\frac{3}{\pi})^{2/3} \frac{h^2}{20 m_e} \rho^{5/3}
\end{aligned} \tag{7}
$$

Numerical solution: $P = 3.83 \times 10^{10}\text{N/m}^2$.

2 Problem 5.14

The **bulk modulus** of substance is the ratio of a small decrease in pressure to the resulting fractional increase in volume:

$$
B = -V \frac{dP}{dV} \tag{8}
$$

Show that $B = (5/3)P$, in the free electron gas model, and use your result in Problem 5.13 (d) to estimate the bulk modulus of copper. *Note*: the observed value is $13.4 \times 10^{10} N/m^2$, but don't expect perfect agreement-after all, we're neglecting all electron-nucleus and electron-electron forces! Actually, it is rather surprising that this calculation comes as close as it does.

2.1 Solution

$$\begin{aligned} B = & -V\frac{dP}{dV} \\ P = & \frac{2}{3}\frac{E_{tot}}{V} \\ \frac{dP}{dV} = & \frac{2}{3}(\frac{E_{tot}}{V})' \\ = & \frac{2}{3}[(\frac{1}{V})'E_{tot} + \frac{1}{V}(E_{tot})'] \\ = & -\frac{2}{3}[\frac{1}{V^2}E_{tot} + \frac{2}{3}\frac{1}{V^2}E_{tot}] \\ = & -\frac{10}{9}\frac{E_{tot}}{V^2} = -\frac{5}{3}\frac{P}{V} \\ B = & (-V)\frac{dP}{dV} = (-V)(-\frac{5}{3}\frac{P}{V}) = \frac{5}{3}P. \end{aligned} \tag{9}$$

numerical solution:$B_{Cu} = 6.383 \times 10^{10}$ J/m^3.

3 Problem 5.17

Suppose we use delta-function *spikes*, instead of *wells* (so that the electrons are *repelled*, instead of attracted, by nuclei). Draw the analog to Figures 5.6 and 5.7 (using the same values of the parameters-except for their signs). How many allowed energies are there in each band? What is the energy at the top of the jth band?

3.1 Solution

Using delta-function *spikes* , the potential will be:

$$V(x) = \alpha\Sigma_{j=0}^{N-1}\delta(x - ja) \tag{10}$$

From the Schrödinger equation and boundary conditions we'll have:

$$f(z) = cos(z) + \beta\frac{sin(z)}{z} \tag{11}$$

For $\beta = 1$, the path will have the shape as shown. There are no negative energies levels, and there are N possibilities of energy in each band, and as z goes to infinity $f(z) = cos(z), cos(z) = 1, z = n\pi$. Since $z = ka, k = n\pi/a$, then $n\pi/a = \sqrt{2mE}/\hbar$. In the end the energy at the top of jth band will be:

$$E_j = \frac{\hbar^2\pi^2 j^2}{2ma^2} \tag{12}$$

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> plot({cos(z)+sin(z)/z,-1,1},z=0...5*Pi);
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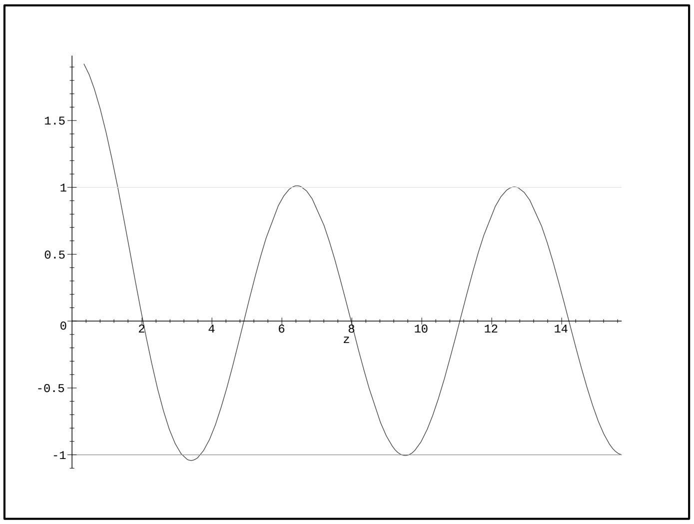

References

[1] D. J. Griffiths, Introduction to Quantum Mechanics, 1995