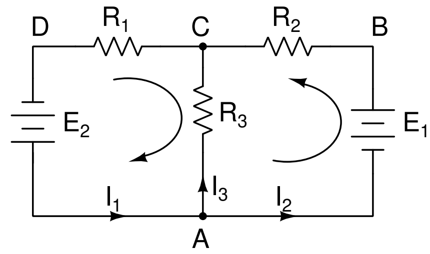

Figure 1: A circuit with two power supplies and three resistors.

Kirchhoff's Laws and Thévenin's Theorem

In this course we will be using a variety of mathematical and conceptual models to describe the electrical components and circuits that we will encounter. Ohm's Law was the first such model. In this chapter we will consolidate some of the concepts from last week into a generic model of a linear device known as a **Thévenin equivalent device (input impedance, output impedance, and internal voltage)**.

The Thévenin equivalents will also be generalized and used to describe properties of signals. A **signal** will be described not only by its **voltage** but also by the **output impedance** of the device that produced or transported it. The concept of the **impedance of a signal** will be used extensively through this course as we discover various methods to transform the properties of a signal.

This week we will also examine network analysis in a complete framework known as Kirchhoff's Laws. These rules will allow us to analyze the properties of any combination of resistors and power supplies.

Kirchhoff's Laws

If you connect lots of power supplies and resistors together in a complicated network, then currents will flow through all the various elements so as to insure that charge is conserved, energy is conserved, and Ohm's Law is satisfied for each resistor. Simultaneously satisfying all these conditions will give you exactly one solution. The method for writing down equations to represent these conservation laws is called Kirchhoff's Laws. To explore these laws, we will consider the sample circuit shown in figure 1.

Kirchhoff's Point Law

Kirchhoff's Point Law says that all of the current that flows into a junction must come out of the junction. This means that charge is conserved – none falls out of the circuit or pools in any location. This concept is illustrated for a junction in the example circuit (see figure 1). In this case, we have

$$I_1 - I_2 - I_3 = 0$$

The original choice of current directions is completely arbitrary. If our solution produce a negative current that means that this current is actually opposite to the selected direction.

Note the sign conventions. We say that a current is positive if we define it to flow into the junction and negative if we define it to flow out the junction.

Kirchhoff's Loop Law

Kirchhoff's Loop Law says that if you sum the voltage drops around any closed loop in a circuit the total must be zero. Since the voltage drops represent the energy per unit charge, this means that all the energy which comes out of the circuit (in heating resistors, for example) must come from some power sources (such as power supplies). This is just conservation of energy.

In our example circuit there are three loops that could be considered (two loops are shown in figure 1 and the third one is the outer loop which we travel clock wise) From these three loops, we obtain the three following relationships:

$$\begin{aligned} V_{DC} + V_{CA} + V_{AD} &= 0 \\ V_{AB} + V_{BC} + V_{CA} &= 0 \\ V_{DC} + V_{CB} + V_{BA} + V_{AD} &= 0 \end{aligned}$$

We can use any two of the above equations (since all three are not independent) and another one is from the point relationship connecting currents. This will give us three independent equations to find three unknowns (currents). To do so we will use two first equation from the loop law.

Our job is to spell out voltage drops through either Ohm's law or via known potential difference provided by batteries. The sign convention is that the voltage drop is positive (increases) if while traveling in chosen direction along the loop we move from negative terminal to a positive terminal of a battery. Similarly, the voltage drop across a resistor is positive when the current flows through a given resistor against direction of our travel. Otherwise, we use negative sign. For our example, this results in

$$\begin{aligned} R_1 \times I_1 + R_3 \times I_3 - E_2 &= 0 \\ E_1 - R_2 \times I_2 + R_3 \times I_3 &= 0 \\ I_1 - I_2 - I_3 &= 0 \end{aligned}$$

All we need to do is to solve this system of equations with known techniques from the linear algebra.

It can be a real nightmare to solve more complicated networks and we will develop models and methods to avoid solving complicated networks wherever we can. We will usually do this by making one part of a circuit relatively independent of another part. Our most common use of Kirchhoff's Laws will usually be to say that $V = IR$ and apply the simple relationships for resistors connected in series or parallel. In the next section we will begin to explore how this is possible.

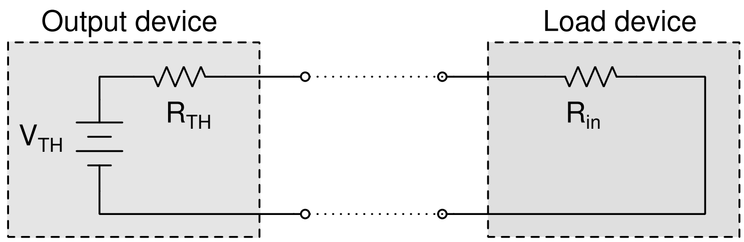

Figure 2: The Thévenin equivalent circuit for an output and an input device.

II. Thévenin Equivalents

The Thévenin equivalents are models to describe the input and output properties of a device. We will assume the input of a device looks like a resistor with the other end connected to ground. We will assume that an output looks like an ideal voltage source followed by a resistor. Both of these are shown in figure 2, below. Note that for this week's discussion the term resistance and impedance are treated as synonymous.

One can determine the input impedance (Z_{in}), which is the same as R_{in} in our example, by simply applying a voltage to the input, measuring the current flowing into the device, and applying Ohm's Law.

Given any electrical device with two output terminals (we will assume one is called ground) there are a number of measurements you might do to try to determine its properties:

1. Measure the voltage across the terminals.
2. Attach a resistor across the terminals and then measure the voltage.
3. Change the resistor and measure the voltage again, repeat.

Clearly, these are all variations on one simple measurement. What is the minimum number of measurements you must do to characterize this black box completely? Thévenin's answer is just two!

Thévenin's Theorem

Any linear (i.e. Ohmic) system with two terminals can be completely characterized as an ideal voltage source (V_{Th} or Thévenin voltage) in series with one resistor (Z_{Th} or Thévenin impedance, and also called internal resistance R_{Th}). No matter what resistance (R_{Load}) you connect across the two terminals, it just forms a voltage divider, as shown in figure 3 on the right, and so the voltage across that load resistor is given by:

$$V_{\mathrm{LOAD}} = V_{\mathrm{TH}} \left(\frac{R_{\mathrm{LOAD}}}{Z_{\mathrm{TH}} + R_{\mathrm{LOAD}}} \right)$$

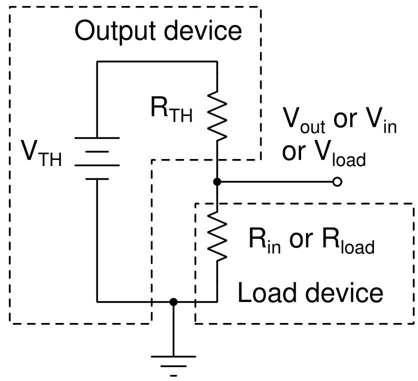

Figure 3: The Thévenin equivalent circuit.

A special case is when there isn't any load applied (i.e. option 1 above, where $R_{LOAD} = \infty$) then the output is simply V_{TH}. From the results of these two measurements we can solve for R_{TH}.

Voltage Divider as a Thévenin Device

Let's consider what happens to our voltage divider from last week when we connect a load resistance to its output. From last week we computed the output of a loaded voltage divider was given by:

$$V_{\mathrm{LOAD}} = V_{\mathrm{IN}} \frac{R_2}{R_1 + R_2} \frac{R_{\mathrm{LOAD}}}{R_{1//2} + R_{\mathrm{LOAD}}}.$$

If we equate factors from in this relationship with the ones in the previous section, we can see that a loaded voltage divider has Thévenin equivalents of

$$V_{\mathrm{TH}} = V_{\mathrm{IN}} \frac{R_2}{R_1 + R_2}$$

and

$$Z_{\mathrm{TH}} = R_{1//2} = \frac{R_1 R_2}{R_1 + R_2}$$

The relationship for V_{TH} makes sense. It is simply the unloaded output voltage as described in the previous section. The output impedance is simply the parallel equivalent of the two resistors on the voltage divider.

Designing a Stiff Voltage Divider

A *stiff* output implies that the output voltage stays relatively close to its Thévenin, or unloaded, voltage when a load is applied. A rule of thumb for designing a stiff voltage source is that the output should not deviate by more than about 10% when loaded. From the relationships in the previous section, we can see that this implies that the load resistance must be more than a factor of 10 larger than the output impedance (in the voltage divider case $R_{TH} = R_{1//2}$). We can think of this at a *Rule of 10.* We will usually design it so that R_1 itself is smaller than the expected load by a factor of 10.

Generalization and Application of the Rule of 10 and Notable Exceptions

Since we are going to design and construct some very complicated circuits this year, we need to be able to focus on parts of circuits, which we will call *sub-circuits.* For example, we will often build amplifiers in three stages. The basic idea is to always ensure that the downstream elements do not load the upstream elements. (1) The first stage will be a "buffer" amplifier that takes a signal and amplifies the current but not the voltage. This will allow us to connect it to the next part of the amplifier without changing (*i.e.* loading) the characteristics of whatever generated the signal. The second stage will be a voltage amplifier that amplifies a signal's voltage to whatever level we need. The final stage will be another buffer amplifier that again amplifies the current so that we will not load our voltage amplifier when we use its output.

This gains us two major advantages: We will never have a large number of interdependent simultaneous equations to solve, and our circuits will work under a wide range of differing operational conditions.

Here is the general procedure:

- Any sub-circuit can be modeled, using Thévenin's Theorem, as an ideal voltage source ($\mathrm{V_{TH}}$) in series with a resistance ($\mathrm{R_{TH}}$).
- When we connect this to the next sub-circuit we can represent that by a load resistance ($\mathrm{R_{LOAD}}$) thus making a voltage divider.
- We can keep our equations simple, so that we need not employ the full Kirchhoff's Laws formalism, simply by following a "factor of 10" rule.

In summary, anything we design to produce a reliable voltage should have $\mathrm{R_{TH}}$ smaller than $\mathrm{R_{LOAD}}$ by least a factor of 10. That is, we do not want the voltage to change significantly when the sub-circuit is connected to subsequent parts of the circuit.

Constant Voltage Sources

The most common sub-circuit we will use will supply a voltage to a later part of a circuit. It may be a power supply to power all the other sub-circuits, or it may be a biasing network to keep a transistor in its normal operating region. For these type sub-circuits, we want V_{OUT} to be relatively independent of $\mathrm{R_{Load}}$, so this means $R_{\mathrm{LOAD}} > 10R_{\mathrm{TH}}$.

Constant Current Sources

We will later see that transistors are current amplifiers. This means that we will often want to drive an amplifier with a current source. A current source can also be modeled as an ideal voltage source in series with an $\mathrm{R_{TH}}$. Again, the complete circuit will look like an ideal voltage source driving a voltage divider, but we want the current through $\mathrm{R_L}$ to be relatively independent of $\mathrm{R_L}$. This means we want the same current as when we remove $\mathrm{R_L}$ and just connect (or "short") the two leads together. By applying the same logic above this implies that we want $R_{\mathrm{LOAD}} < 10R_{\mathrm{TH}}$ for a stiff *current source*.

Impedance Matching

There are two cases when we require $R_{\mathrm{LOAD}} = R_{\mathrm{TH}}$, which we will call "impedance matching". If you want the most efficient transfer of power to your load, then you must choose $R_{\mathrm{LOAD}} = R_{\mathrm{TH}}$. By the most efficient transfer, we mean that power received and dissipated by load

$$I^2 R_L = \left(\frac{V_{in}}{R_L + R_{Th}}\right)^2 R_L$$

will have its largest value when $R_{\mathrm{LOAD}} = R_{\mathrm{TH}}$. Note that V_{in} is input voltage from the load point of view, and this is the same as V_{out} from the output of the signal or power source point of view.

Of course this will severely load the source, dropping the output voltage by exactly a factor of two. This may, however, be fine in simple circuits – like lighting a light-bulb or driving an audio speaker.

The second case is when you have very fast signals traveling along transmission cables such as the BNC cables that we often use to connect laboratory equipment. Then, we will want to prevent reflections at our connections, since these could give rise to phony signals. Then we will want to choose $R_{\mathrm{LOAD}} = R_{\mathrm{TH}}$, even though it will decrease the amplitude of the signal. We often accomplish this matching by attaching a 50Ω *terminator* to the end of our BNC cable to ensue that the impedance is matched.