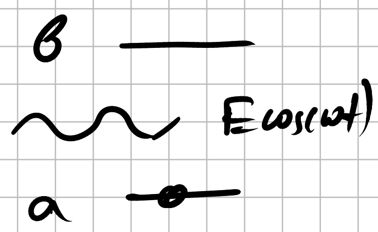
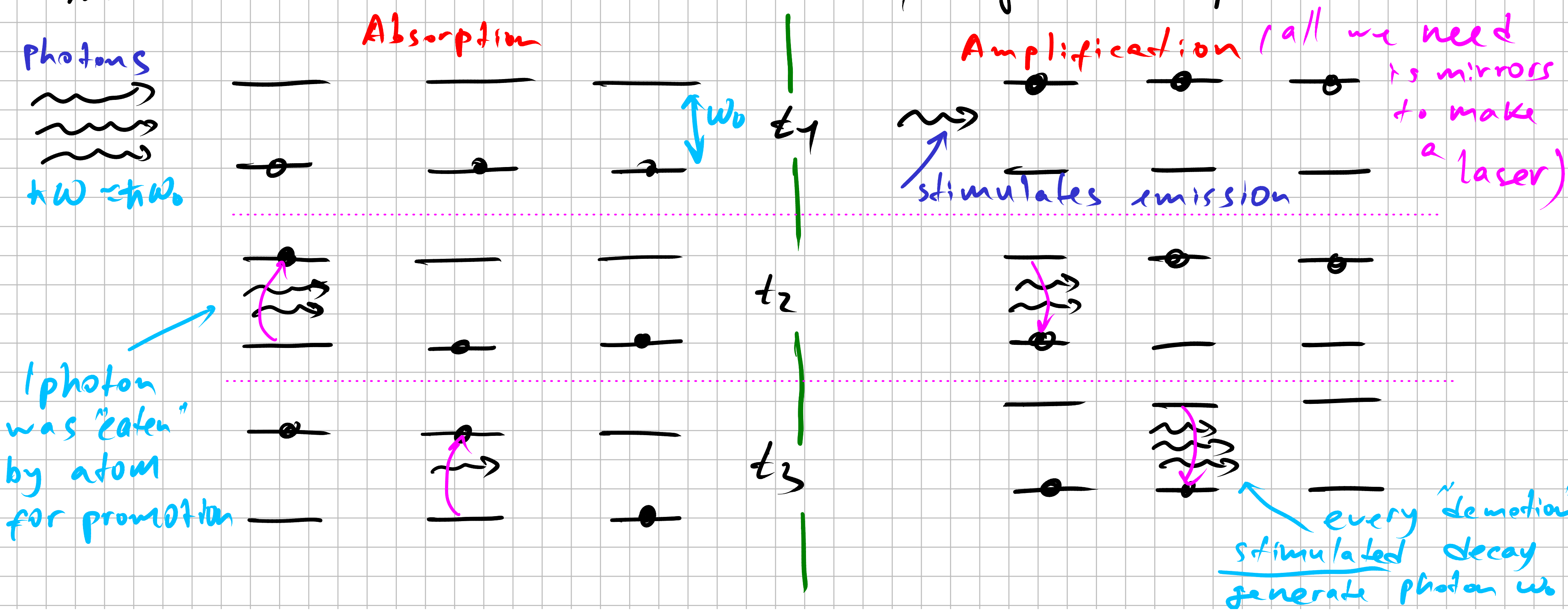
Two-level atom spontaneous emission (decay)



1st note that by swapping $a \leftrightarrow b$ nothing changes in our equations

so if we found probability to "promote" (excite) from level $a \to b$ it is the <u>same</u> as to decay from $b \to a$ (if we start at 'b')

Now we can understand idea of light absorption



In either case we **must** have a changing in time Hamiltonian, otherwise $c_n \Psi_n$

$|c_n|^2$ becomes constant!

So QM states in general case

$\hat{H}$ independent of time $\Rightarrow |c_n|^2 = \text{constant}$

(remember about $c_n e^{-iE_n t/\hbar}$ phase term)

However experimentally we see that exited (higher energy) states decay down usually we blame external fields since there is no way to have an atom isolated.

But actually the reason is more fundamental: there are vacuum fluctuations even if you deplete everything from excess energy make a light tight box there is $\frac{1}{2}\hbar\omega$ field fluctuations

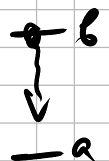
ground energy of an oscillator

Quantatization of light is subject of Quantum ElectroDynamics (QED), so we just take a note that EM wave consist of photons $\hbar\omega$ and can be treated as a harmonic oscillator.
The stronger (brighter) the light the more energetic it is i.e. more photons in it. But even pitch black situation carries $\frac{1}{2}\hbar\omega$ vacuum fluctuations

So high energy decay is not spontaneous but simulated by those vacuum fluctuations

We will attempt to estimate lifetime of an exited state somewhat simplisticly.

1st recall probability of stimulated transition



$$P_{b\to a}^{(\omega)} = \left(\frac{V_{ab}}{\hbar} \frac{\sin\left(\frac{\omega_0-\omega}{2}t\right)}{\omega_0-\omega} \right)^2$$

why we are not using exact case? B/s we have solution only for $\omega_0 = \omega$ (resonant case)

recall $V_{ab} = E\langle a | q\hat{x} | b\rangle$
$= E\wp$ ← dipole element, more pricesely polarization thus curly $p \to \wp$

Those vacuum fluctuations are incoherent
$\Rightarrow$ happen at all frequencies with no phase relationship

so
$$P_{b\to a}(t) = \int_0^\infty d\omega \left(\frac{E(\omega)\wp}{\hbar} \cdot \frac{\sin\frac{\omega_0-\omega}{2}t}{\omega_0-\omega} \right)^2$$

For monochromatic field $\quad \rho(\omega) = \dfrac{\varepsilon_0 E^2(\omega)}{2}$

energy density of field i.e. how much energy carried by given ω field

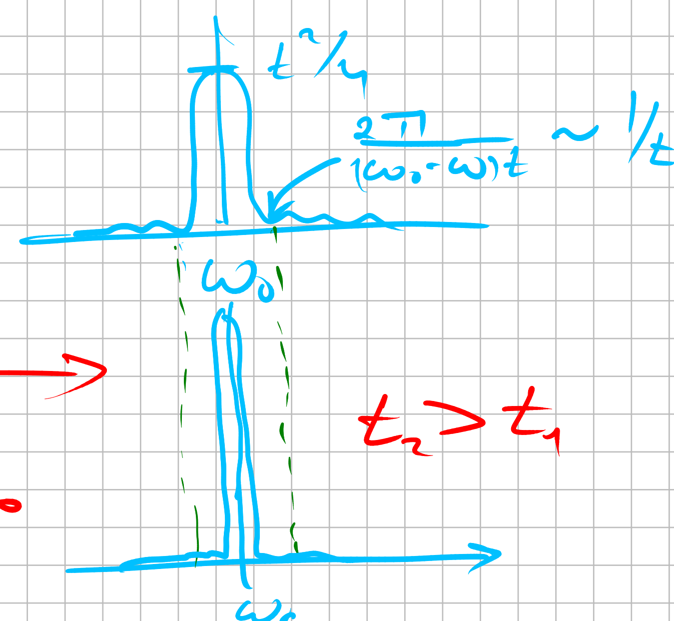
$$P_{b\to a}(t) = \int \frac{2}{\varepsilon_0}\rho(\omega)\left(\frac{\wp}{\hbar}\right)^2 \left(\frac{\sin\frac{(\omega_0-\omega)}{2}t}{\omega_0-\omega}\right)^2 d\omega$$

$$\left(\frac{\sin\frac{(\omega_0-\omega)}{2}t}{(\omega_0-\omega)/2}\right)^2 \xrightarrow[\omega_0\to\omega]{} \frac{t^2}{4} \Longrightarrow$$

$$\parallel$$

$$\frac{\pi}{2}\delta((\omega_0-\omega)t) = \frac{\pi}{2}t\,\delta(\omega_0-\omega)$$

resembles δ function for $t\to\infty$



$$P_{b\to a}(t) = \frac{\pi}{\varepsilon_0}\left|\frac{\wp}{\hbar}\right|^2 \rho(\omega_0)\, t$$

We will be concerned with Black Body radiation which is everywhere and comes from all directions

$$\wp = q\,\vec{r}\cdot\vec{n}$$

direction of polarization

$\vec{r} \parallel \vec{z}$

$$= q\, z \cdot \cos\theta$$

$$\langle \wp^2 \rangle = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} \left| \langle a | qz | b \rangle \cos\theta \right|^2 \cdot \sin\theta\, d\theta$$

$$= \frac{1}{3}\left|\langle a | qz | b\rangle\right|^2 = \frac{1}{3}\wp_{ab}$$

average about all solid angles

full solid angle

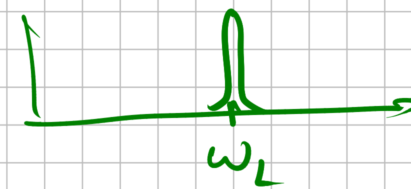
$$\Rightarrow \frac{dP_{b\to a}}{dt} = \boxed{\frac{\pi}{3\varepsilon_0}\left|\frac{\wp_{ab}}{\hbar}\right|^2 \rho(\omega_0) = B_{b\to a}\,\rho(\omega_0)}$$

Rate of transition in a unit of time

Some examples of $\rho(\omega)$

laser (very close to monochromatic)

$$\rho(\omega) = \frac{\rho_0\, \gamma^2}{(\omega-\omega_L)^2 - \gamma^2}$$



Black body radiation very broad occupies all ω

Now imagine many - many atoms
some are in state 'a' (N_a of them)
the others in state 'b' (N_b)

in a given time some of them spontaneously
decay with rate 'A' $\Rightarrow \frac{dN_b}{dt} = -A \cdot N_b$

b

more atoms in state b, more decays we see

some will be simulated down

a

with $-B_{ba} \cdot \rho(\omega_0) N_b$ some promoted up
with $+B_{ab}\, \rho(\omega_0) N_a$

Einstein coefficient

$$\Rightarrow \frac{dN_b}{dt} = -A N_b - B_{ba}\, \rho(\omega_0) N_b + B_{ab}\, \rho(\omega_0) N_a \qquad (*)$$

Black Body radiation is constant in equilibrium
thus number of decays $\frac{dN_b}{dt} = 0$ otherwise we amplify the field

$$\Rightarrow \rho(\omega_0) = \frac{A \cdot N_b}{B_{ab} N_a - B_{ba} N_b} = \frac{A}{\frac{N_a}{N_b} B_{ab} - B_{ba}}$$

Stat Mechanics says Boltzmann distribution $\frac{N_a}{N_b} = \frac{e^{-E_a/kT}}{e^{-E_b/kT}} = e^{\hbar\omega_0/kT}$

$$\rho(\omega_0) = \frac{A}{e^{\hbar\omega_0/kT} B_{ab} - B_{ba}}$$

We can compare it with Black Body radiation density (spectrum)

$$\rho_{BB}(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/kT} - 1} = \rho(\omega_0)$$

$$\Rightarrow \quad B_{ab} = B_{ba} \text{ and } = \frac{\pi}{3\varepsilon_0}\left|\frac{\wp_{ab}}{\hbar}\right|^2$$

we already know it

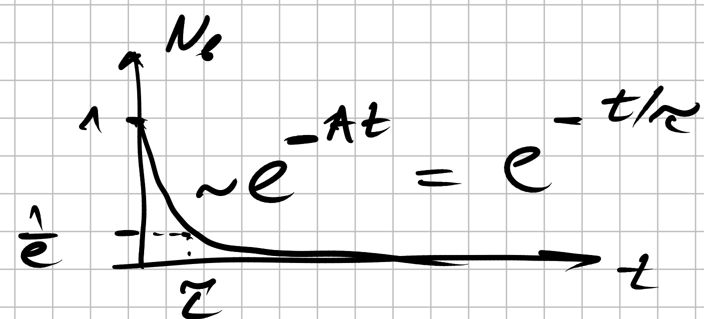
$$\frac{A}{B_{ab}} = \frac{\hbar\omega_0^3}{\pi^2 c^3} \Rightarrow \boxed{A = \frac{\hbar\,\omega_0^3}{3\pi c^3 \varepsilon_0} |\wp_{ab}|^2}$$

What is the meaning of 'A'.

Let's look at (*) and think about single atom: once photon is emited there is nothing to simulate and there are no photons from outside $\Rightarrow \rho(\omega_0) = 0$

$$\Rightarrow \frac{dN_b}{dt} = -A N_b$$

$$N_b = \underset{\substack{\| \\ 1}}{N_b(0)}\, e^{-A \cdot t}$$



so life time of exited state $\boxed{\tau = 1/A}$

Note $A \sim |\wp_{ab}|^2 = |q \langle a_{nlm} | z | b_{n'l'm'} \rangle|^2$

recall that it $\neq 0$ for $\Delta l = 1$, $\Delta m = \pm 1, 0$

the stronger the matrix element the stronger the transition

i.e. chance of spontaneous decay but thus of simulated transition

Optics Jargon