

Scattering

The Born approximation

- Integral form of solution for Schrödinger equation

$$(\nabla^2 \psi + k^2 \psi) = \frac{2m}{\hbar^2} V \psi = Q, \quad k^2 = \frac{2mE}{\hbar^2}$$

(*) $\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{2m}{\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} V(\vec{r}') \psi(\vec{r}') d^3\vec{r}'$, where ψ_0 satisfies source free equation $(\nabla^2 + k^2) \psi_0(\vec{r}) = 0$

Let's focus on solution in far zone $r \rightarrow \infty$

Assuming $V(\vec{r}') \xrightarrow{r' \rightarrow \infty} 0$ i.e. we are interested in case $r' \ll r$

we approximate

$$\begin{aligned} |\vec{r} - \vec{r}'| &= \sqrt{(\vec{r} - \vec{r}')^2} = \sqrt{r^2 - 2\vec{r} \cdot \vec{r}' + r'^2} \\ &\approx \sqrt{r^2 \left(1 - \frac{2\vec{r} \cdot \vec{r}'}{r^2}\right)} = r \sqrt{1 - \frac{2\hat{r} \cdot \hat{r}'}{r}} \\ &\approx r - \hat{r} \cdot \vec{r}' \end{aligned}$$

unit vector
 $\hat{r} = \frac{\vec{r}}{r}$

Let's define $\vec{k} = k \cdot \hat{r} \rightarrow (x)$ we keep this it is oscillating term

$$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2} \int \frac{e^{i\vec{k}\vec{r} - i\vec{k}'\vec{r}'}}{|\vec{r} - \vec{r}'|} V(\vec{r}') \psi(\vec{r}') d^3\vec{r}'$$

we neglect small correction,

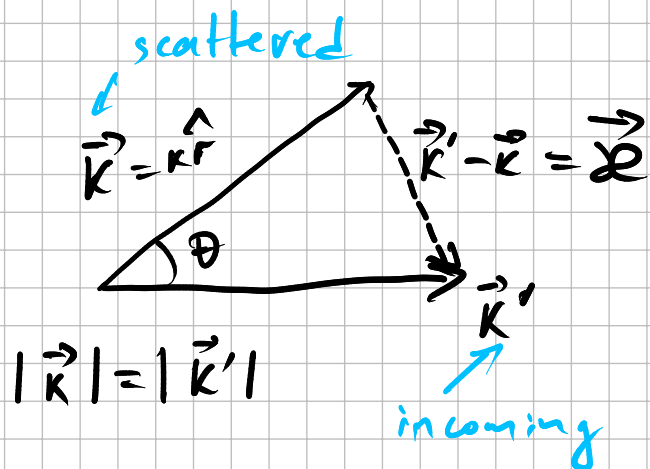
$$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2} \frac{e^{i\vec{k}\vec{r}}}{r} \int e^{-i\vec{k}'\vec{r}'} V(\vec{r}') \psi(\vec{r}') d^3\vec{r}'$$

recall $\psi(\vec{r}) = A \left(e^{i\vec{k}\vec{z}} + f(\theta) \frac{e^{i\vec{k}\vec{r}}}{r} \right)$

$$\Rightarrow f(\theta) = -\frac{m}{2\pi\hbar^2} \frac{1}{A} \int e^{-i\vec{k}'\vec{r}'} V(\vec{r}') \psi(\vec{r}') d^3\vec{r}'$$

Born approximation

$$\psi(\vec{r}') \approx \psi_0(\vec{r}') = A e^{i\vec{k}'\vec{z}'} = A e^{i\vec{k}'\vec{r}'}$$



weak potential
almost no change
to incoming wave

$$\vec{k}' = k \hat{z}$$

3D Fourier transform

$$f(\theta) = -\frac{m}{2\pi\hbar^2} \int e^{i(\underbrace{\vec{r}' - \vec{r}}_x) \cdot \vec{r}'} V(\vec{r}') d^3\vec{r}'$$

Spherically symmetric potential $V(\vec{r}) = V(r)$

$$f(\theta) = -\frac{m}{2\pi\hbar^2} \int e^{i\vec{x}\vec{r}'} V(r') r'^2 \sin(\theta') d\theta' d\phi' dr'$$

$$\vec{x}\vec{r}' = x r' \cos \theta' \quad (\text{we direct } z' \text{ along } \vec{x})$$

$$f(\theta) = -\frac{m}{2\pi\hbar^2} 2\pi \int e^{i x r' \cos \theta'} r'^2 dr' \cdot d(-\underbrace{\cos \theta'}_{\pm}) V(r')$$

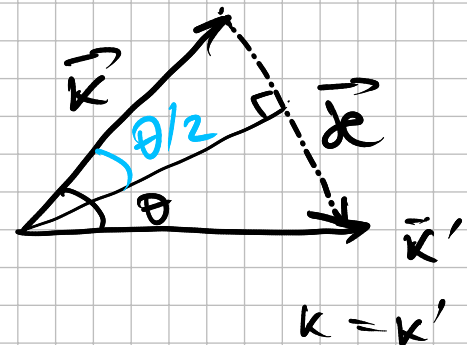
$$= -\frac{m}{2\pi\hbar^2} \int r'^2 dr' \left(-\frac{e^{i x r' \pm}}{i x r'} \right) V(r') =$$

$$= \frac{m}{\hbar^2} \int \frac{r' dr'}{i x} \underbrace{(e^{-i x r'} - e^{i x r'})}_{-2i \sin(x r')} V(r') =$$

$$f(\theta) = -\frac{2m}{\hbar^2} \frac{1}{x} \int_0^\infty \sin(x r') r' V(r') dr'$$

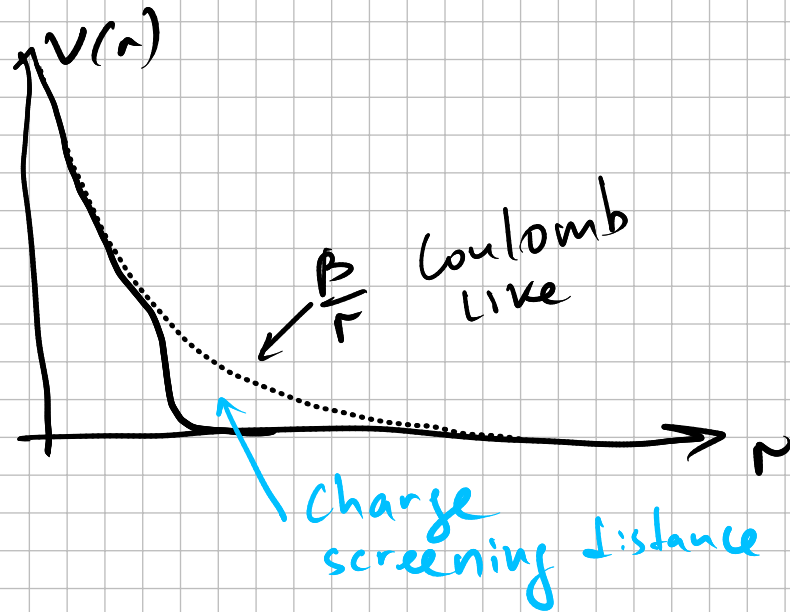
where is θ dependence in $\uparrow$?

$$x = 2k \sin(\theta/2)$$



Example: Yukawa potential

$$V(r) = \beta \frac{e^{-\mu r}}{r}$$



$$f(\theta) = -\frac{2m}{\hbar^2 \chi} \beta \int_0^\infty e^{-\mu r} \sin(\chi r) dr$$

$r' \rightarrow r$

$$\frac{e^{i\chi r} - e^{-i\chi r}}{2i}$$

$$= -\frac{2m}{\hbar^2 \chi} \frac{\beta}{2i} \left[\frac{e^{(i\chi - \mu)r}}{i\chi - \mu} - \frac{e^{(-i\chi - \mu)r}}{-i\chi - \mu} \right] \Big|_0^\infty$$

$$= -\frac{2m}{\hbar^2 \chi} \frac{\beta}{2i} \left[\frac{1}{i\chi - \mu} + \frac{1}{i\chi + \mu} \right]$$

$$f(\theta) = -\frac{2m}{\hbar^2} \frac{\beta}{\chi^2 + \mu^2}$$