Identical particles:

* Distinguishable
* Fermions
* Bosons

Suppose we have two identical particle,
i.e. we cannot tell them apart, they are indistinguishable

$\Rightarrow \quad \hat{H} = \hat{H}_{1,2}$

If you blink and someone switches them,
the physical word is the same: $\hat{H}_{1,2} = \hat{H}_{2,1}$
also probability density must stay the same

$$|\psi(\vec{r}_1, \vec{r}_2)|^2 = |\psi(\vec{r}_2, \vec{r}_1)|^2$$

$$\Rightarrow \boxed{\psi(\vec{r}_1, \vec{r}_2) = \pm \psi(\vec{r}_2, \vec{r}_1)}$$

We can introduce exchange operator $\hat{P}$ (do not confuse it with parity!)

$$\hat{P}\,\psi(\vec{r}_1,\vec{r}_2) = \psi(\vec{r}_2,\vec{r}_1)$$

Clearly $\hat{P}\hat{P}\,\psi(\vec{r}_1,\vec{r}_2) = \psi(\vec{r}_1,\vec{r}_2) \Rightarrow \hat{P}\,\lambda\psi(\vec{r}_2,\vec{r}_1) = \lambda^2\psi(\vec{r}_1,\vec{r}_2)$

eigen value of $\hat{P}$

$$\lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

symmetric wavefunction

$$\Rightarrow \hat{P}\,\psi(\vec{r}_1,\vec{r}_2) = \boxed{\psi(\vec{r}_2,\vec{r}_1) = \pm 1\,\psi(\vec{r}_1,\vec{r}_2)}$$

anti-symmetric

same as above conclusion

One more property.

$$\boxed{[\hat{P},\hat{H}] = 0}$$

Easy to prove if we recall that $H_{1,2} = H_1 + H_2 = H_{2,1}$

Thus if state of the system is symmetric or anti-symmetric, it will stay the same. !

Particles which form symmetric states combination we call bosons,

and in anti-symmetric state — fermions

Note that we can have a compound particle

$$\psi(1,2,3,\dots N) = +\psi(2,1,3\dots) = +\psi(2,3,1,\dots) = \psi_+$$

boson

$$\text{or} \quad = -\psi(2,1,3,\dots) = +\psi(2,3,1,\dots) = \psi_-$$

fermion

sign flips for pair wise exchange

But because $[\hat{M}, \hat{p}] = 0$

once a boson — alway a boson

fermion — - - - fermion

By some "strange" reason bosons have integer spin and fermions $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$

We need relativistic QM to prove it.

Note you can stick couple fermions together and make bosonic particle, if the total spin is integer

Let's try to construct a simplest boson or fermion $\psi_{\pm}(\vec{r}_1, \vec{r}_2)$

1st a little refresher: ψ function of two non-interacting particles

$$\psi(\vec{r}_1, \vec{r}_2), \quad \hat{H} = \underbrace{-\frac{\hbar^2}{2m_1}\nabla_1^2 + V_1(\vec{r}_1)}_{H_1} \underbrace{-\frac{\hbar^2}{2m_2}\nabla_2^2 + V_2(\vec{r}_2)}_{H_2}$$

We will seek for solution in the form $\psi_a(\vec{r}_1) \cdot \psi_b(\vec{r}_2)$

$$\hat{H}\,\psi(\vec{r}_1, \vec{r}_2) = \hat{H}\,\psi_a(\vec{r}_1) \cdot \psi_b(\vec{r}_2) =$$

$$= \big(H_1\,\psi_a(\vec{r}_1)\big) \cdot \psi_b(\vec{r}_2) + \big(H_2\,\psi_b(\vec{r}_2)\big) \cdot \psi_a(\vec{r}_1) =$$

$$= E\,\psi_a(\vec{r}_1) \cdot \psi_b(\vec{r}_2) = \big(E_1\,\psi_a(\vec{r}_1)\big)\psi_b + \big(E_2\,\psi_b(\vec{r}_2)\big)\psi_a(\vec{r}_1)$$

$\Rightarrow$ separation of variables

$$\left\{\begin{array}{l} H_1\,\psi_a(\vec{r}_1) = E_1\,\psi_a(\vec{r}_1) \\ H_2\,\psi_b(\vec{r}_2) = E_2\,\psi_b(\vec{r}_2) \end{array}\right\} \text{2 independent Schrödinger equations}$$

Important result: we just proved that the eigen state of two independent particles could be constructed as a product of two independent states

$$\psi(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1) \cdot \psi_b(\vec{r}_2)$$

same true for N independent particles

$$\psi(\vec{r}_1, \dots, \vec{r}_N) = \psi_a(\vec{r}_1)\,\psi_b(\vec{r}_2) \cdot \dots \psi_z(\vec{r}_N)$$

Now! the interesting part

$\psi_\pm$ it cannot be just $\psi_a(r_1) \cdot \psi_b(r_2)$

since $\hat{P}\,\psi_a(\vec{r}_1)\,\psi_b(\vec{r}_2) \neq \pm\,\psi_a(\vec{r}_2)\,\psi_b(\vec{r}_1)$

but we can construct a linear combination

$$\boxed{\psi_\pm(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}}\Big(\psi_a(\vec{r}_1)\,\psi_b(\vec{r}_2) \pm \psi_a(\vec{r}_2) \cdot \psi_b(\vec{r}_1)\Big)}$$

Check yourself that $\hat{P}\,\psi_\pm(1,2) = \pm\,\psi(1,2)$

Note: $\hat{P}$ switches either ψ_i, ψ_k or $\vec{r}_i, \vec{r}_k$ subindex not both!

We all heard many times that fermions cannot take the same state. Let's see why is it the case from exchange point of view

Fermion $\Rightarrow$ $\psi_- = \frac{1}{\sqrt{2}}\left(\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) - \psi_a(\vec{r}_2)\psi_b(\vec{r}_1) \right)$

now we put both particles to the same state 'a'

$$\psi_-(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}}\left(\psi_a(\vec{r}_1)\psi_a(\vec{r}_2) - \psi_a(\vec{r}_2)\psi_a(\vec{r}_1) \right)$$

if we apply exchange operator we expect

$$\hat{P}\psi_-(\vec{r}_1, \vec{r}_2) = \psi_-(\vec{r}_2, \vec{r}_1) = -\psi_-(\vec{r}_1, \vec{r}_2)$$

but in our case $\hat{P}\psi_-(\vec{r}_1, \vec{r}_2) = \psi_-(\vec{r}_2, \vec{r}_1) = +\psi_-(\vec{r}_1, \vec{r}_2)$

so we arive to contradiction

our assymetric wavefunction becomes symmetric!

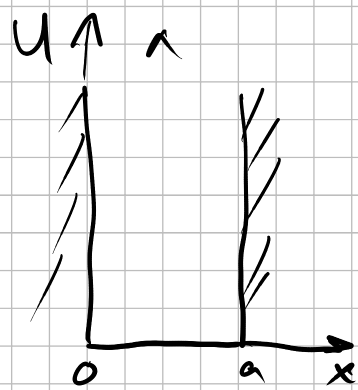
Wait a second, we all know that we can have 2 electron in for example 's' ($\ell = 0$) orbital.

Yes, but they have different spin projection.

Their states are not exactly the same.

For a while we will forget about spin!
We just care about is it fermions or bosons

Let's put **two non-interacting** particles in the square well



For every particle

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \quad , \quad E_n = n^2 E_1$$

We have 3 cases

distinguishable particles

$E_g = 2E_1$

$\psi_g = \psi_1(x_1)\cdot\psi_2(x_2) =$

$= \frac{2}{a}\sin\left(\frac{\pi}{a}x_1\right)\cdot\sin\left(\frac{\pi}{a}x_2\right)$

$\psi_{fe} = \psi_1(x_1)\,\psi_2(x_2)$

or ... $\psi_2(x_1)\,\psi_1(x_2)$

degenerate

$E_{fe} = 5E_1$

identical bosons

$E_g = 2E_1$

$\psi_g = \frac{1}{\sqrt{2}}\left(\psi_1(x_1)\psi_1(x_2) + \psi_1(x_2)\psi_1(x_1)\right)$

$\psi_{fe} = \frac{1}{2}\left(\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2)\right)$

non degenerate

$E_{fe} = 5\,E_1$

identical fermions

There is no level !!! with $E_g = 2E_1$ ...

since it would 2 fermions in the same state

The $E_g = 5E_1$

$\psi_g(x_1,x_2) = \frac{1}{\sqrt{2}}\left(\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)\right)$

$E_{fe} = (1+9)E_1, \quad \psi_{fe} = \frac{1}{\sqrt{2}}\left(\psi_1(x_1)\psi_3(x_2) - \psi_3(x_1)\psi_1(x_2)\right)$