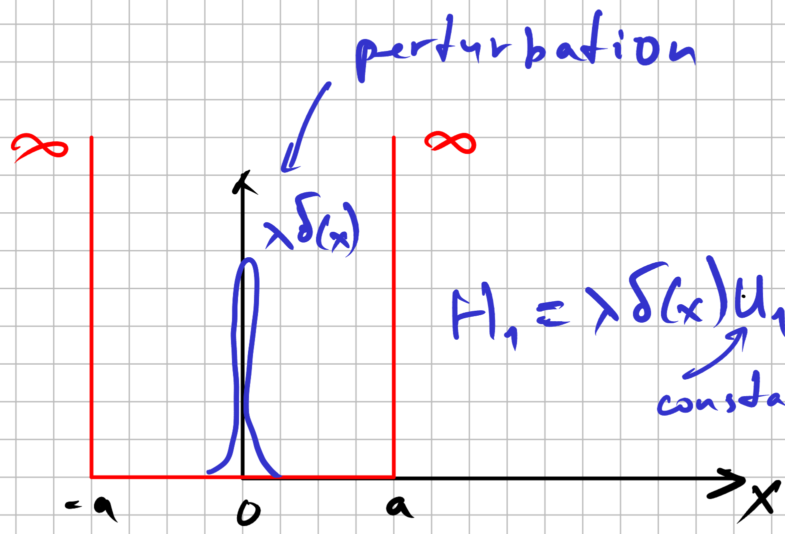
Example: Square well + δ in the midle



$H_1 = \lambda\delta(x) U_1$

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + U_0(x)$$

$$U_0 = \begin{cases} 0, & x < |a| \\ \infty, & x > |a| \end{cases}$$

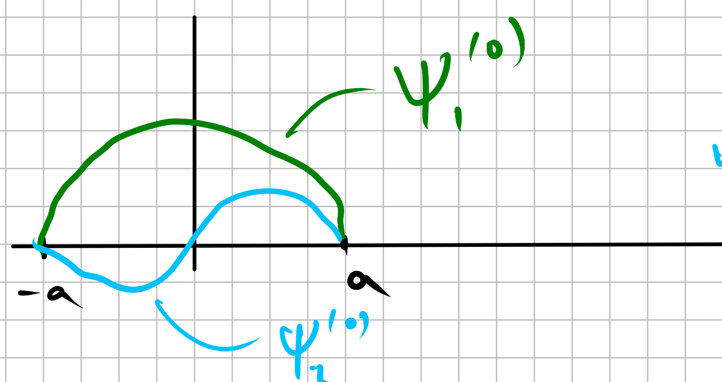
$$\psi_n^{(0)} = \sqrt{\frac{1}{a}} \times \begin{cases} \cos\left(\frac{\pi}{2a} n x\right), & n = 1, 3, 5, \ldots \\ \sin\left(\frac{\pi}{2a} n x\right), & n = 2, 4, 6, \ldots \end{cases}$$

for $x < |a|$

$\psi_n^{(0)}(|x| > a) = 0$

Boundary condition

$$E_n^{(0)} = \frac{\hbar^2}{2m}\left(\frac{\pi \cdot n}{2a}\right)^2$$



note that $\psi_n^{(0)}$ with even 'n' has zero value at $x = 0$

i.e. seems to not interact with $H_1 = \lambda\delta(x)$

let's confirm that $E_n^{(1)}$ for $n=2,4,6..$ (even) is zero.

$$E_n^{(1)} = \langle \psi_n^{(0)} | H_1 | \psi_n^{(0)} \rangle =$$

$$= \int_{-a}^{\infty} \sqrt{\tfrac{1}{a}} \begin{matrix}\cos\\ \sin\end{matrix} \left(\frac{\pi}{2a} n x \right) \lambda \delta(x) U_1 \cdot \sqrt{\tfrac{1}{a}} \begin{matrix}\cos\\ \sin\end{matrix} \left(\frac{\pi}{2a} n x \right) dx =$$

$$= \lambda \frac{1}{a} \times \begin{matrix}\cos^2\\ \sin^2\end{matrix} \left(\frac{\pi}{2a} n \cdot 0 \right) = \lambda \frac{1}{a} \times \begin{pmatrix} U_1 \\ 0 \end{pmatrix}$$

given by cos $n=1,3,5$

given by sin, $n=2,4,6..$

$$E_n^{(1)} = \begin{cases} \frac{\lambda U_1}{a}, & n=1,3,5.. \\ 0, & n=2,3,5 \end{cases}$$

What about $c_{kn} = \dfrac{\langle \psi_k | H_1 | \psi_n \rangle}{E_n^{(0)} - E_k^{(0)}}$, since sin-like ψ do not overlap with H_1 we expect no change in $\psi_{n=2,4,6}$

$$\langle \psi_k | H_1 | \psi_n \rangle = \int_{-a}^{a} dx \sqrt{\frac{1}{a}} \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{\pi}{2a} n x\right) \cdot \lambda U_1 \delta(x) \cdot \sqrt{\frac{1}{a}} \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{\pi}{2a} k x\right)$$

$$= \frac{1}{a} \lambda U_1 \begin{matrix}\cos\\ \sin\end{matrix}(0) \cdot \begin{matrix}\cos\\ \sin\end{matrix}(0) = \begin{cases} \frac{1}{a}\lambda U_1, & n \in 1,3,5\ldots \text{ and } k \in 1,3,5\ldots \\ 0 & \text{if } k \text{ or } n \text{ is even} \end{cases}$$

Note that even $E_n^{(2)}$ for even n is zero