Time independent perturbation with degenerate levels.

Note that we generally need a terms

Like $\dfrac{\langle \psi_k^{(0)} | H_1 | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_k^{(0)}}$

but if we have degeneracy $E_n = E_k$ the above term <u>explodes</u>, since we have $\frac{1}{0}$

$\Rightarrow$ we need <u>different</u> strategy to handle perturbation of degenerate levels

<u>Side note:</u> we cannot even do 1st order correction

$E_n^{(1)} = \langle \psi_n | H_1 | \psi_n \rangle$

since if $\psi_a^{(0)}, \psi_b^{(0)}, \psi_c^{(0)} \dots$ are eigenstates of $\hat{H}_0$ with the same eigen value $E_n^{(0)}$ than any linear combination is eigenstate too

$|\psi_n^{(0)}\rangle = c_a |\psi_a^{(0)}\rangle + c_b |\psi_b^{(0)}\rangle + c_c |\psi_c^{(0)}\rangle + \dots$

but different set of 'c' form different $|\psi_n\rangle$

But we can reuse old approach with small modifications. We are solving:

$$(\hat{H}_0 + \lambda \hat{H}_1)\,\varphi_n = E_n \varphi_n =$$

as before $\quad E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} \dots$

$$\varphi_n = \varphi_n^{(0)} + \lambda \varphi_1^{(1)} + \lambda^2 \varphi^{(2)} + \dots$$

Focusing only on terms $\sim \lambda$ we have

$$\hat{H}_0 |\psi_n^{(1)}\rangle + \hat{H}_1 |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(1)}\rangle + E_n^{(1)} |\psi_n^{(0)}\rangle \qquad *$$

so far no surprises we saw it before

Here is the difference, 'n' state is degenerate

$$\Rightarrow \quad \psi_n^{(0)} = c_a |\psi_a^{(0)}\rangle + c_b |\psi_b^{(0)}\rangle + c_c |\psi_c^{(0)}\rangle \dots = c_i |\psi_i^{(0)}\rangle$$

we drop 'n' subindex, otherwise notation is to bulky. But we must remember about $\hat{H}_0 |\psi_{a,b,c\dots}\rangle = E_n |\psi_{a,b,c\dots}\rangle$

Let's calculate $\langle \psi_a^{(0)} | * \rangle$

$$\cancel{\langle \psi_a^{(0)} | H_0 | \psi_n^{(1)}\rangle} + \langle \psi_a | H_1 | \psi_n^{(0)}\rangle = \cancel{\langle \psi_a^{(0)} | E_0^{(0)} | \psi_n^{(1)}\rangle} + \langle \psi_a^{(0)} | E_n^{(1)} | \psi_n^{(0)}\rangle$$

$(\langle \psi_a^{0} | \hat{H}_0^{+} = E_n^{(0)} \langle \psi_a^{(0)} |$

$$\langle \psi_a | H_1 | \psi_n^{(0)} \rangle = E_n^{(1)} \langle \psi_a^{(0)} | \psi_n^{(0)} \rangle = E_n^{(1)} \cdot c_a$$

recall $\psi_n^{(0)} = c_a|\psi_a\rangle + c_b|\psi_b\rangle + c_c|\psi_c\rangle + \cdots$

$$c_a \underbrace{\langle \psi_a | H_1 | \psi_a \rangle}_{H_{1aa}} + c_b \underbrace{\langle \psi_a | H_1 | \psi_b \rangle}_{H_{1ab}} + c_c \underbrace{\langle \psi_a | H_1 | \psi_c \rangle}_{H_{1ac}} + \cdots = E_n^{(1)} \cdot c_a$$

We can generalize

$$\langle \psi_m | \times \rangle \Rightarrow c_a H_{1ma} + c_b H_{1mb} + c_c H_{1mc} + \cdots = E_n^{(1)} c_m$$

or in matrix notation

$$\underbrace{\begin{pmatrix} H_{1aa} & H_{1ab} & H_{1ac} & \cdots \\ H_{1ba} & H_{1bb} & H_{1bc} & \cdots \\ H_{1ca} & H_{1cb} & H_{1cc} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}}_{\tilde{H}_1} \begin{pmatrix} c_a \\ c_b \\ c_c \\ \vdots \end{pmatrix} = E_n^{(1)} \begin{pmatrix} c_a \\ c_b \\ c_c \\ \vdots \end{pmatrix}$$

$$\boxed{\tilde{H}_1 \cdot \underset{\downarrow}{c} = E_n^{(1)} \cdot \underset{\downarrow}{c}}$$

$\underset{\downarrow}{c}$ is eigenvector
$E_n^{(1)}$ is eigenvalue

$\hat{H}_1$ in basis of 'n'th level degenerate states slightly rearanging above equation

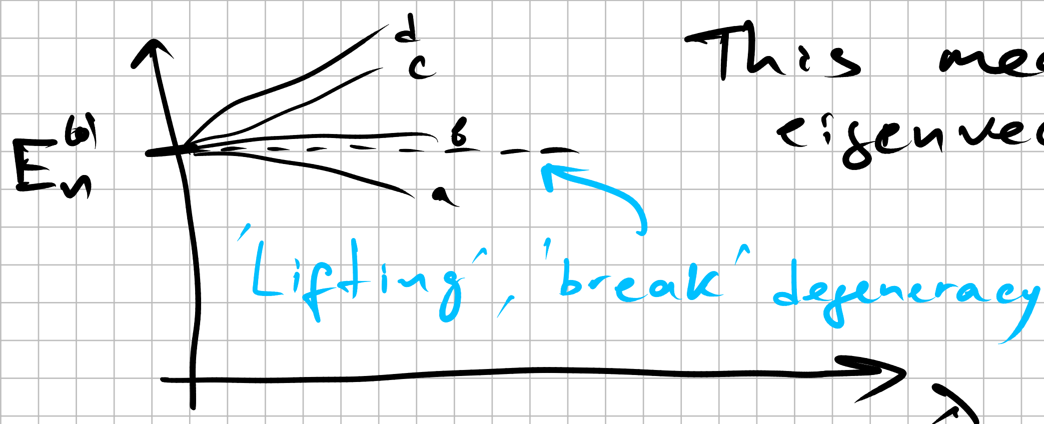
$$\underbrace{\begin{pmatrix} H_{1aa}-E_n^{(1)} & H_{1ab} & H_{1ac} & \cdots \\ H_{1ba} & H_{1bb}-E_n^{(1)} & H_{1bc} & \cdots \\ H_{1ca} & H_{1cb} & H_{1cc}-E_n^{(1)} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}}_{(\tilde{H}_1 - E_n^{(1)}\tilde{I})} \begin{pmatrix} c_a \\ c_b \\ c_c \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

identity $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ & & 1 \end{pmatrix}$

characteristic equation

to find $E_n^{(1)}$ we solve $\boxed{\det(\tilde{H}_1 - E_n^{(1)}\tilde{I}) = 0}$

in general there are as many solutions as is degeneracy of the 'n' level



This means several eigenvectors $c_\downarrow$ as well!

If off diagonal terms of $\tilde{H}_1 = 0$,
i.e $\tilde{H}_1$ is diagonal then it is easy to find solution an 'C' combinations

$$\det \begin{pmatrix} H_{1aa} - E_n^{(1)} & & \bigcirc \\ & H_{1bb} - E_n^{(1)} & \\ \bigcirc & & H_{1cc} - E_n^{(1)} \\ & & \end{pmatrix} = 0$$

$$(H_{1aa} - E_n^{(1)})(H_{1bb} - E_n^{(1)})(H_{1cc} - E_n^{(1)}) \ldots = 0$$

$$\Rightarrow \quad E_n^{(1)} = H_{1aa}, \quad H_{1bb}, \quad H_{1cc}, \ldots$$

$$\underset{\downarrow}{C} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ldots$$

There are gazillion ways to express any of constituting degenerate states, e.g. ψ_a

it would be lovely to "guess" a set of $\psi_a, \psi_b, \psi_c \ldots$ so $\tilde{M}_1$ is diagonal.

Unfortunately, there is no <u>constructive</u> way to do it, before solving "characteristic" equation.

We can search for states of operator $\hat{O}$ which commute with $\hat{H}_1$ and $\hat{H}_0$, then in $|0\rangle$ basis our $\tilde{M}_1$ matrix will be diagonal. But this shift the question to "how to find such $\hat{O}$?" which in general not easy.