Let's find 2nd correction to Energy

$$(\hat{H}_0 + \lambda \hat{H}_1)(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \lambda^2|\psi_n^{(2)}\rangle + \bar{O}(\lambda^3)) =$$
$$= (E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \bar{O}(\lambda^3))(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \lambda^2|\psi_n^{(2)}\rangle + \bar{O}(\lambda^3))$$

we will worry only about λ^2 terms:
lower order (0) and (1) we already did and
λ^3 is not our concern.

$$\boxed{\lambda^2 \times (\hat{H}_0|\psi_n^{(2)}\rangle + \hat{H}_1|\psi_n^{(1)}\rangle) = \lambda^2 \times (E_n^{(0)}|\psi_n^{(2)}\rangle + E_n^{(1)}|\psi_n^{(1)}\rangle + E_n^{(2)}|\psi_n^{(0)}\rangle) \quad *}$$

Our goal to find $E_n^{(2)}$, so we will calculate $\langle \psi_n^{(0)} | * \rangle$

$$\underbrace{\langle \psi_n^{(0)}|H_0}_{\langle \psi_n^{(0)}|E_n^{(0)}}|\psi_n^{(2)}\rangle + \langle \psi_n^{(0)}|H_1|\psi_n^{(1)}\rangle = E_n^{(0)}\langle \psi_n^{(0)}|\psi_n^{(2)}\rangle + E_n^{(1)}\underbrace{\langle \psi_n^{(0)}|\psi_n^{(1)}\rangle}_{=0} + E_n^{(2)}\underbrace{\langle \psi_n^{(0)}|\psi_n^{(0)}\rangle}_{1=}$$

$$E_n^{(2)} = \langle \psi_n^{(0)}|H_1|\psi_n^{(1)}\rangle$$

since $|\psi_n^{(1)}\rangle = \sum_{k\neq n} c_{kn}|\psi_k^{(0)}\rangle$
and $\langle \psi_n^{(0)}|\psi_k^{(0)}\rangle = 0$

$$E_n^{(2)} = \langle \psi_n^{(0)} | H_1 | \psi_n^{(1)} \rangle =$$

$$= \langle \psi_n^{(0)} | H_1 | \left(\sum_{k\neq n} c_{kn} |\psi_k^{(0)}\rangle \right) =$$

$$= \langle \psi_n^{(0)} | H_1 | \left(\sum_{k\neq n} \underbrace{\frac{\langle \psi_k^{(0)} | H_1 | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_k^{(0)}}}_{\text{just a number}} |\psi_k^{(0)}\rangle \right) =$$

$$= \sum_{k\neq n} \frac{\langle \psi_n^{(0)} | H_1 | \psi_k^{(0)} \rangle \cdot \langle \psi_k^{(0)} | H_1 | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_k^{(0)}}$$

note $\langle \psi_n | H | \psi_k \rangle = \left(\langle \psi_k | H | \psi_n \rangle \right)^*$

since H is Hermitian

$$\boxed{E_n^{(2)} = \sum_{k\neq n} \frac{|\langle \psi_n^{(0)} | H_1 | \psi_k^{(0)} \rangle|^2}{E_n^{(0)} - E_k^{(0)}}}$$

Note: $E^{(2)}_{ground} \leq 0$ <u>always</u>!

since $E_g < E_n$