

Time dependent perturbations

So far we were focusing on time independent Schrödinger equation $\hat{H} \neq \hat{H}(t)$

$$i\hbar \frac{d\psi}{dt} = \hat{H} \psi \quad \text{which we obtain by doing variable separation } \psi(\vec{r}, t) = e^{-\frac{iEt}{\hbar}} \psi(\vec{r})$$
$$\Rightarrow \hat{H} \psi(\vec{r}) = E \psi(\vec{r})$$

So general solution

$$\text{is } \psi(\vec{r}, t) = \sum_n e^{-\frac{iE_n t}{\hbar}} c_n \cdot \psi_n(\vec{r}) \quad (*)$$

\vec{r} could be anything
(n, l, m, \dots) but time

boring phase factor

Now we add time dependence and split Hamiltonian to $\hat{H}_0 + \hat{H}_1(t)$

let's assume we can solve

$$\hat{H}_0 \psi_n^{(0)} = E_n^{(0)} \psi_n^{(0)}$$

, then $\{\psi_n^{(0)}\}$ is the full set and any function can be expressed as $\sum_n c_n \psi_n^{(0)}$

this is true for $\psi(t, \vec{r})$: $\hat{H} \psi(t, \vec{r}) = (\hat{H}_0 + \hat{H}_1(t)) \psi(t, \vec{r}) = i\hbar \frac{d}{dt} \psi(t, \vec{r})$

$$\psi(t, \vec{r}) = \sum_n c_n(t) e^{-iE_n^{(0)}t/\hbar} \psi_n^{(0)}(\vec{r})$$

comparing with $(*)$ c_n is time dependent!

for convenience we drop \vec{r} : $\psi(t, \vec{r}) \rightarrow \psi(t)$, $\psi_n^{(0)}(\vec{r}) \rightarrow \psi_n^{(0)}$

$$\begin{aligned} \Rightarrow (\hat{H}_0 + \hat{H}_1) |\psi(t)\rangle &= \sum_n c_n(t) e^{-\frac{iE_n^{(0)}t}{\hbar}} (\hat{H}_0 + \hat{H}_1) |\psi_n^{(0)}\rangle = \\ &= i\hbar \frac{d}{dt} |\psi(t)\rangle = i\hbar \sum_n \left(\frac{d}{dt} c_n(t) - \frac{iE_n^{(0)}}{\hbar} c_n(t) \right) e^{-iE_n^{(0)}t/\hbar} |\psi_n^{(0)}\rangle \\ &= \sum_n c_n(t) e^{-iE_n^{(0)}t/\hbar} (E_n^{(0)} + \hat{H}_1) |\psi_n^{(0)}\rangle \end{aligned}$$

let's "multiply" by $\langle \psi_f^{(0)} |$ and recall $\langle \psi_f^{(0)} | \psi_n^{(0)} \rangle = \delta_{fn}$

$$\frac{d}{dt} c_f(t) = \frac{-i}{\hbar} \sum_n c_n(t) e^{\frac{i(E_f^{(0)} - E_n^{(0)})t}{\hbar}} \langle \psi_f^{(0)} | \hat{H}_1(t) | \psi_n^{(0)} \rangle$$

this exact but complicated
since $\sum_n c_n$, contains c_f

We will do perturbation trick

and assume \hat{H}_1 small $\Rightarrow \hat{H}_1 \rightarrow \lambda \hat{H}_1$

$$c_n(t) = c_n^{(0)} + \lambda c_n^{(1)} + \lambda^2 c_n^{(2)} \dots, \quad \lambda \ll 1$$

Collecting terms $\sim \lambda^0$ we see

$$\boxed{\frac{d}{dt} c_f^{(0)} = 0}, \quad \text{since } \langle \psi_f^{(0)} | \lambda \hat{H}_1 | \psi_n^{(0)} \rangle \sim \lambda \dots$$

Next we assume that at initial time (usually 0 or $-\infty$)
the system "starts" at energy level 'i' (we will use 0 here)
and then $H_1(t)$ is "on"

$$\Rightarrow c_n^{(0)} = \delta_{ni} \quad \text{and} \quad c_n^{(k)}(0) = 0, \quad \text{for any } k \geq 1$$

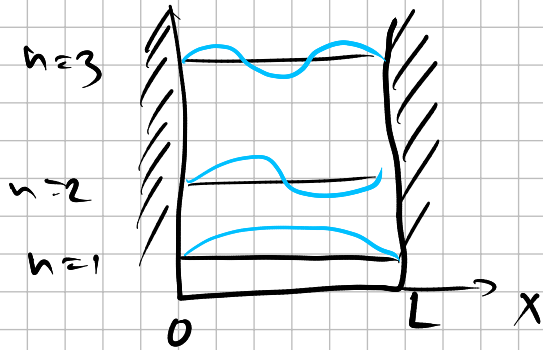
let's focus on λ^1 term

$$\begin{aligned} \frac{d}{dt} c_f^{(1)}(t) &= -\frac{i}{\hbar} \sum_n \underbrace{c_n^{(0)}(t)}_{\delta_{ni}} e^{i(E_f - E_n)t/\hbar} \langle \psi_f^{(0)} | H_1 | \psi_n^{(0)} \rangle \\ &= -\frac{i}{\hbar} e^{i(E_f - E_i)t/\hbar} \langle \psi_f^{(0)} | H_1 | \psi_i^{(0)} \rangle \end{aligned}$$

$$\Rightarrow c_f(t) = \delta_{fi} - \frac{i}{\hbar} \int_0^t e^{i(E_f - E_i)t'/\hbar} \langle \psi_f^{(0)} | \hat{H}_1(t') | \psi_i^{(0)} \rangle dt'$$

$c_f^{(0)}$, here we did $\lambda H_1 \rightarrow H_1$
 "back conversion"

Example:

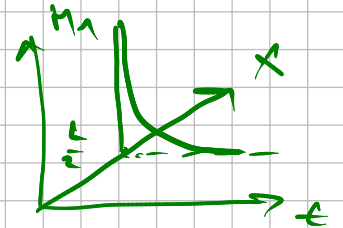


$$\psi_n^{(0)} = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right)$$

$$H_0 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$$

$$V(x) = \begin{cases} 0, & 0 \leq x \leq L \\ \infty, & \text{elsewhere} \end{cases}$$

$$M_1(t) = 2 \delta\left(x - \frac{L}{2}\right) e^{-t/\tau}$$



let's set initial state to be $n=1$

kroncker, do not confuse with

$$c_f(t) = \delta_{f1} - \frac{i}{\hbar} \int_0^t dt' e^{i(E_f - E_1)t'/\hbar} \int_0^L dx \frac{2}{L} \sin\left(\frac{\pi}{L} f x\right) \cdot \sin\left(\frac{\pi}{L} x\right) 2 \delta\left(x - \frac{L}{2}\right) e^{-t'/\tau}$$

$$= \delta_{f1} - \frac{i}{\hbar} \int_0^t dt' e^{i(E_f - E_1)t'/\hbar} \frac{2d}{L} \underbrace{\sin\left(\frac{\pi}{2} f\right) \sin\left(\frac{\pi}{2}\right)}_{\neq 0 \text{ only for odd } f} e^{-t'/\tau}$$

$$= \delta_{f1} - \frac{i 2d}{L} \sin\left(\frac{\pi}{2} f\right) \left[\frac{e^{-\frac{t}{\tau} + i(E_f - E_1)t/\hbar} - 1}{-\frac{1}{\tau} + i \frac{(E_f - E_1)}{\hbar}} \right]$$

$$(E_f - E_i)/\hbar = \omega_{fi} \quad \text{oscillation frequency}$$

$$C_f(t) = \delta_{f1} - \frac{i2d}{L} \sin\left(\frac{\pi}{2}f\right) \frac{e^{-t/\tau} + i\omega_{fi}t - 1}{-\frac{1}{\tau} + i\omega_{fi}}$$

Probability to be detected in state $f = 2m+1 \neq 1$

$$P_{2m+1} = \left| \frac{2d}{L} \right|^2 \frac{(e^{-t/\tau} \cdot e^{i\omega_{fi}t} - 1)(e^{-t/\tau} e^{-i\omega_{fi}t} - 1)}{\left(\frac{1}{\tau}\right)^2 + \omega_{fi}^2}$$

$$\begin{aligned} & (e^{-t/\tau} e^{i\omega_{fi}t} - 1)(e^{-t/\tau} e^{-i\omega_{fi}t} - 1) = \\ & = e^{-2t/\tau} + 1 - e^{-t/\tau} \underbrace{(e^{i\omega_{fi}t} + e^{-i\omega_{fi}t})}_{2\cos(\omega_{fi}t)} \end{aligned}$$

$$P_{f=2m+1} = \left| \frac{2d\tau}{L} \right|^2 \frac{1 + e^{-\frac{2t}{\tau}} - 2\cos(\omega_{fi}t)e^{-t/\tau}}{1 + (\omega_{fi}\tau)^2}$$

for very long z , and $t \ll z$,

$$P_{f=2m+1} = \left| \frac{2z\tau}{L} \right|^2 \frac{4 \sin^2\left(\frac{\omega_{f1}t}{2}\right)}{1 + (\omega_{f1}z)^2}$$

$t \gg z$

$$P_{f=2m+1} = \left| \frac{2z\tau}{L} \right|^2 \frac{1}{1 + (\omega_{f1}z)^2}$$

