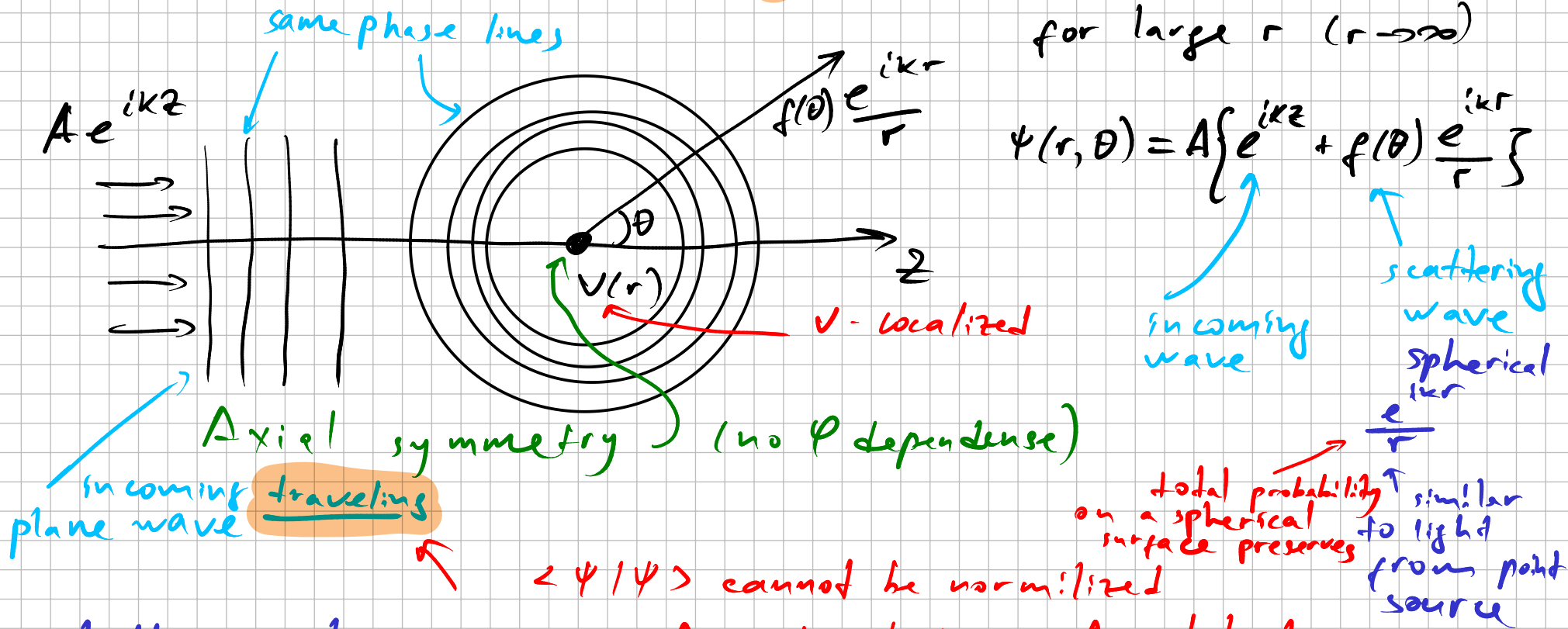


# Quantum scattering



I find the easiest  
 to apply my  
 intuition of  
 the light scattering  
 to see similarity  
 (when  $V=0$ )

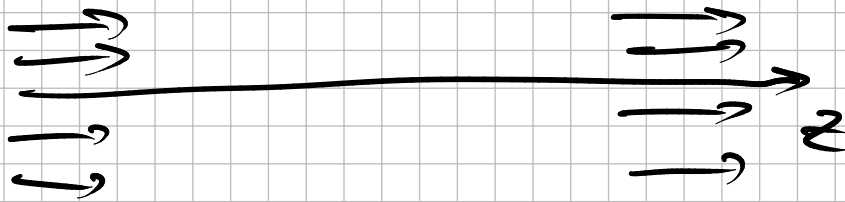
eq. of motions  
 are the same:  $(\nabla^2 + k^2)\psi = 0$

Think about laser pointer  $\Rightarrow$   
 it goes to infinity and has  $\infty$  number  
 of photons  $\Rightarrow$   $\infty$  Energy. This is unphysical  
 so we talk about local Amplitude  
 and not normalization.

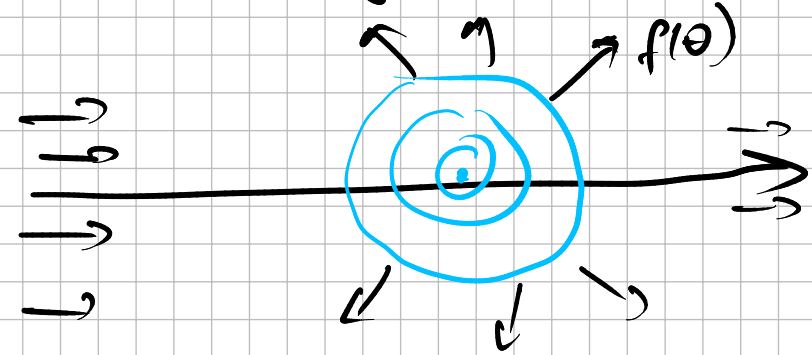
Why this paradox? We assumed  
 infinite time of observation

Compare two cases

no scattering potential  
nothing scatters



scattering potential



so  $f(\theta)$  proportional to probability to find a scattered particle at angle  $(\theta)$  or  $(\Omega)$

Differential cross section

$$D(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

In QM we are concerned with finding  $f(\theta) \Rightarrow \sigma$

$$\sigma = \int |f(\theta)|^2 d\Omega$$

in experiment we can access  $|f(\theta)|$  and we want to know  $\sigma$  and  $V(r)$

What is our justification for

$$\psi = A \left( e^{ikz} + f(\theta) e^{ikr} \right)$$

↑ this force particular shape (plane wave) of incoming way. i.e. we assume we know it.

$$H\psi = E\psi$$

$$\frac{\hat{p}^2}{2m} + V(r) = E\psi$$

$$\left( -\frac{\hbar^2}{2m} \nabla^2 + V(r) \right) \psi = E\psi(r, \theta, \varphi)$$

we assume symmetry so  $\varphi$  is irrelevant  $\Rightarrow m=0$

$$\psi(r, \theta) = R(r) Y_l^m(\theta, \varphi)$$

$$u = r \cdot R(r)$$

$$r \rightarrow \infty \Rightarrow V(r) \rightarrow 0$$

$$\approx -\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left[ \underset{0}{V(r)} + \frac{\hbar^2}{2m} \frac{\cancel{l(l+1)}}{\cancel{r^2}} \right] = E u \quad (*)$$

$\rightarrow 0 \quad r \rightarrow \infty$

this follow from

$$\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \frac{2m r^2}{\hbar^2} [V(r) - E] R = l(l+1) R$$

$r \rightarrow \infty$   
 $\Rightarrow$

$$-\frac{\hbar^2}{2m} \frac{d^2 u(r)}{dr^2} = E u(r)$$

$$u(r) = C e^{ikr}$$

outgoing wave

$$+ D e^{-ikr}$$

incoming wave

we force discussion to outgoing wave only (scattering)

Combining all together

$$\psi(r, \theta) = A \left( e^{ikz} + f(\theta) \frac{e^{ikr}}{r} \right)$$

$$\sim \frac{u}{r} = R(r)$$

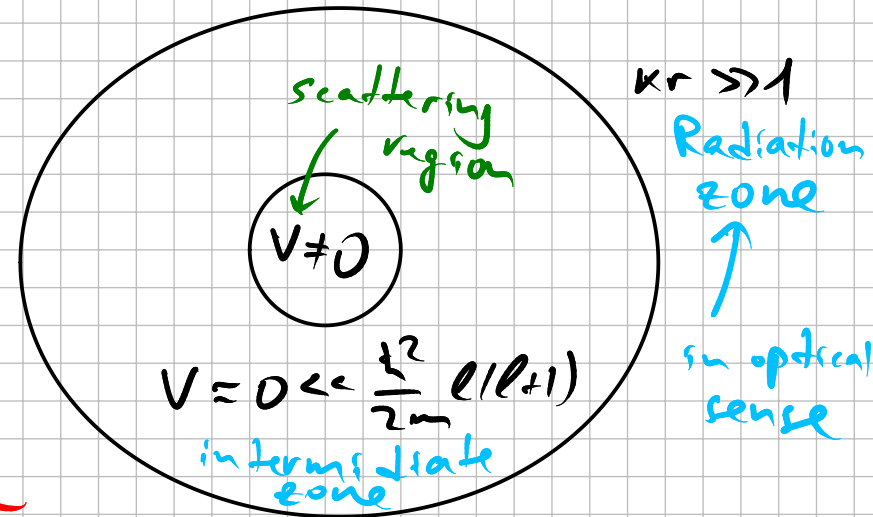
# Partial wave analysis

## Intermediate zone

$$V(r) \ll \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2}$$

↑  
we can neglect it  
in intermediate zone

Note electrostatic potential  $\sim 1/r$   
does not satisfy this condition!



from (\*)

$$\frac{d^2 u}{dr^2} - \frac{l(l+1)}{r^2} u = -\frac{2mE}{\hbar^2} u = -k^2 u$$

solution

$$u(r) = A r j_l(kr) + B r n_l(kr)$$

some times called  
spherical Neumann  
function

spherical Bessel function  
analogous to 1D

sin, cos in 1D

similar to  $e^{i\varphi} = \cos \varphi + i \sin \varphi$

we can write

$$h_e^{(1)} \equiv j_e(x) + i n_e(x) ; \quad h_e^{(2)} \equiv j_e(x) - i n_e(x)$$

spherical Hankel functions of  
the first and second kind

$$h_0^{(1)} = -i \frac{e^{ix}}{x}$$

$$h_1^{(1)} = \left(-\frac{i}{x^2} - \frac{1}{x}\right) e^{ix}$$

$$h_2^{(1)} = \left(-\frac{3i}{x^3} - \frac{3}{x^2} + \frac{i}{x}\right) e^{ix}$$

$$h_0^{(2)} = i \frac{e^{-ix}}{x}$$

$$h_1^{(2)} = \left(\frac{i}{x^2} - \frac{1}{x}\right) e^{-ix}$$

$$h_2^{(2)} = \left(\frac{3i}{x^3} - \frac{3}{x^2} - \frac{i}{x}\right) e^{-ix}$$

---

$$x \rightarrow \infty ; h_e^{(1)} = \frac{1}{x} (-i)^{l+1} e^{ix} ; \quad h_e^{(2)} = \frac{1}{x} (i)^{l+1} e^{-ix}$$

So  $\Psi(r, \theta)$  can be thought as  $A' h_e^{(1)}(kr) + B' h_e^{(2)}(kr)$

but  $h_e^{(2)} \sim e^{-ixr}$  and thus  $B' = 0$   
ingoing wave

## Extra note on Math Physics

$$j_\ell(x) = (-x)^\ell \left( \frac{1}{x} \frac{d}{dx} \right)^\ell \frac{\sin x}{x} \Rightarrow j_0(x) = \frac{\sin x}{x}$$

$$n_\ell(x) = -(-x)^\ell \left( \frac{1}{x} \frac{d}{dx} \right)^\ell \frac{\cos x}{x} \Rightarrow n_0(x) = -\frac{\cos x}{x}$$

for  $x \rightarrow 0$

$$j_\ell(x \approx 0) = \frac{2^\ell \ell!}{(2\ell+1)} x^\ell$$

$$n_\ell(x \approx 0) = \frac{(2\ell)!}{2^\ell \ell!} \frac{1}{x^{\ell+1}}$$

So most general solution  $R(r) \sim \sum A'_l h_l^{(1)}(r)$   
 $l=m=0$  no  $\varphi$  dependence

$$\Psi(r, \theta) = A \left( e^{ikz} + \sum_l A'_l h_l^{(1)}(kr) Y_l^0(\theta, \varphi) \right)$$

Recall  $Y_l^0(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$   
 $\uparrow$  Legendre polynomial

$$\Rightarrow \Psi(r, \theta) = A \left\{ e^{ikz} + \kappa \sum_{l=0}^{\infty} i^{l+1} (2l+1) a_l h_l^{(1)}(kr) P_l(\cos\theta) \right\}$$

note we redefined  $A_l = i^{l+1} \kappa \sqrt{4\pi(2l+1)} a_l$

$a_l$  - called partial wave amplitude

You may have heard s-wave ( $l=0$ )  
 p-wave ( $l=1$ ) etc.

Why such complicated form?

for  $r \rightarrow \infty$ :  $h_l(kr) = (-1)^{l+1} e^{ikr} / kr$

$$\Psi(r, \theta) = A \left\{ e^{ikz} + \frac{e^{ikr}}{r} \sum_{l=0}^{\infty} (2l+1) a_l P_l(\cos\theta) \right\} = A \left\{ e^{ikz} + f(\theta) \frac{e^{ikr}}{r} \right\}$$



One more simplification

orthonormal  
see'

$$\sigma = D(\theta) d\Omega = \int |f(\theta)|^2 d\Omega = \frac{1}{4\pi} \sum_e \sum_{e'} (2\ell+1)(2\ell+1) a_e^* a_{e'} P_e(\cos\theta) P_{e'}(\cos\theta) d\Omega$$

$\int_0^\pi \int_0^{2\pi} r^2 \sin\theta d\phi d\theta$

$$\sigma = 4\pi \sum_e (2\ell+1) |a_e|^2$$

$$\int_{-1}^1 P_\ell(x) P_{\ell'}(x) dx = \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

We may wonder why we keep  $(2\ell+1)$  in front of  $a_e$  (we could have absorbed it into  $a_e$ )?

It helps to simplify the following:

note that we use  $z$  and  $r, \theta$  which seems not needed since  $z = r \cos(\theta)$  or  $r \cos(\pi)$

There is **Rayleigh's formula**:

$$e^{ikz} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta)$$

$$\psi(r, \theta) = A \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) \{ j_\ell(kr) + ik a_\ell h_\ell^{(1)}(kr) \} P_\ell(\cos\theta)$$

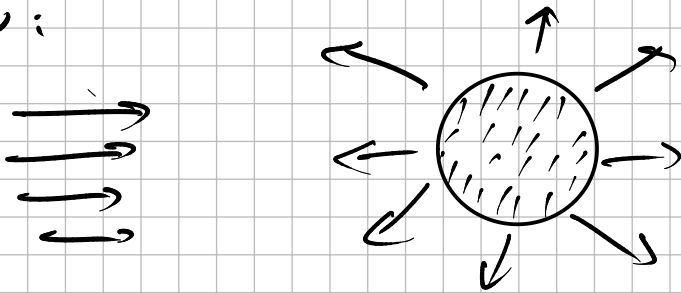
In region where we neglect  $V(r)$

see  
(xx)  
for start  
point

Example: hard-sphere scattering

$$V(r) = \begin{cases} \infty, & r < a \\ 0, & r \geq a \end{cases}$$

2D view:



Boundary condition

$$\Psi(a, \theta) = 0$$

$$\Psi(r, \theta) = A \sum_{l=0}^{\infty} i^l (2l+1) \left\{ j_l(kr) + i k a e h_l^{(1)}(kr) \right\} P_l(\cos \theta)$$

at first it seems hard since  $\sum_l$  is involved

Observe  $\int P_{l'}(\cos \theta) \Psi(a, \theta) d\Omega = 0$

$l'$  prime = Const.  $\cdot \{ \dots \} \delta_{l'l} = 0 \Rightarrow \{ \dots \} = 0$

$$\Rightarrow j_l(ka) + i k \cdot a e h_l^{(1)}(ka) = 0$$

$$\Rightarrow a_e = \frac{i}{k} \frac{j_l(ka)}{h_l^{(1)}(ka)}$$

Scattering amplitude depends on  $k = \frac{2\pi}{\lambda}$

and thus Energy

Classically this was not the case!

let's consider the case of long-wave / low-energy scattering  
 $\lambda \gg a \Leftrightarrow ka \ll 1$

$$a_l = \frac{i}{k} \frac{j_l(ka)}{h_l^{(+)}(ka)} = \frac{i}{k} \frac{j_l(ka)}{\cancel{j_l(ka)} + i n_l(ka)}$$

small compared to  $n_l$  for  $ka \ll 1$

for  $x \rightarrow 0$

$$j_l(x \approx 0) = \frac{2^l l!}{(2l+1)} x^l$$

$$n_l(x \approx 0) = \frac{(2l)!}{2^l l!} \frac{1}{x^{l+1}}$$

$$a_l = \frac{i}{k} \frac{1}{2l+1} \frac{(2^l l!)^2}{(2l)!} (ka)^{2l+1}$$

for  $ka \ll 1$

recall 
$$\sigma = 4\pi \sum_{l=0}^{\infty} (2l+1) |a_l|^2$$

drops like a rock as  $l$  goes up  
 so we keep only  $l=0$

$$\sigma \approx 4\pi a^2$$

wow!

4 larger than classical case!

Full area of spheres  $\psi$  "feels" it